An unknotting theorem for
delta and sharp edge-homotopy

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Abstract

Two spatial embeddings of a graph are said to be delta (resp. sharp) edge-homotopic if they are transformed into each other by self delta (resp. sharp) moves and ambient isotopies. We show that any two spatial embeddings of a graph are delta (resp. sharp) edge-homotopic if and only if the graph does not contain a subgraph which is homeomorphic to the theta graph or the disjoint union of two 1-spheres, or equivalently $G$ is homeomorphic to a bouquet.

Keywords: Spatial graph; Delta move; Sharp move; $C_k$-move

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1. Introduction

Let $G$ be a finite graph which does not contain a free vertex. We consider $G$ as a topological space in the usual way. An embedding $f : G \to S^3$ is called a spatial embedding of $G$ or simply a spatial graph. A graph $G$ is said to be planar if there exists an embedding $G \to S^2$. For a planar graph $G$, a spatial embedding of $G$ is said to be trivial if it is ambient isotopic to an embedding $G \to S^2 \subset S^3$. We note that a trivial spatial embedding of a planar graph is unique up to ambient isotopy in $S^3$ [6].

In the following we recall the three equivalence relations on spatial graphs generated by specific local moves as follows:

(1) A crossing change is a local move on a spatial graph as illustrated in Fig. 1.1. A crossing change is called a self crossing change if all two strings in the move belong to the same spatial edge. Two spatial embeddings of a graph are said to be edge-homotopic \(^1\) if they are transformed into each other by self crossing changes and ambient isotopies.

(2) A delta move [7], [11] is a local move on a spatial graph as illustrated in Fig. 1.2. A delta move is called a self delta move if all three strings in the move belong to the same spatial edge. Two spatial embeddings of a

\(^1\) This equivalence relation was called simply a homotopy in [23].
graph are said to be *delta edge-homotopic* if they are transformed into each other by self delta moves and ambient isotopies.

(3) A *sharp move* [10] is a local move on a spatial oriented graph as illustrated in Fig. 1.3. A sharp move is called a *self sharp move* if all four strings in the move belong to the same spatial edge. Two spatial embeddings of a graph are said to be *sharp edge-homotopic* if they are transformed into each other by self sharp moves and ambient isotopies. If we turn the orientations of all strings in a self sharp move the other way at once, then the concluded move is also a self sharp move. Therefore this equivalence relation does not depend on the edge orientations.

Fig. 1.1.

![Fig. 1.1](image1.png)

Fig. 1.2.

![Fig. 1.2](image2.png)

Fig. 1.3.

![Fig. 1.3](image3.png)

Edge-homotopy on spatial graphs was introduced by Taniyama in [23] as a generalization of *link homotopy* in the sense of Milnor [8]. Delta edge-homotopy and sharp edge-homotopy on spatial graphs were introduced by the author in [15] and [17] as generalizations of *self $\Delta$-equivalence* [22] (or *delta link homotopy* [12]) and *self $\sharp$-equivalence* [20] on oriented links, respectively. It is known that the implication $(2) \Rightarrow (3) \Rightarrow (1)$ holds [17, Theorem 1.1].
Obviously the crossing change is an *unknotting operation*, namely every knot can be undone by crossing changes and ambient isotopies. Besides it is well known that the delta move and the sharp move are also unknotting operations [7], [11], [10]. In general, if any two spatial embeddings of a graph \( G \) are transformed into each other by a finite sequence of specific local moves and ambient isotopies, then the local move is called a *uniforming operation* for the spatial embeddings of \( G \) [18]. We have already known when the self crossing change is a uniforming operation as follows:

**Theorem 1.1.** ([23, Theorem B]) For a graph \( G \), the following are equivalent.

1. Any two spatial embeddings of \( G \) are edge-homotopic.
2. \( G \) does not contain a subgraph which is homeomorphic to \( K_4 \), \( D_3 \) or the disjoint union of two 1-spheres, where \( K_4 \) and \( D_3 \) are graphs as illustrated in Fig. 1.4.
3. \( G \) is a generalized bouquet, namely there exists a vertex \( v \) of \( G \) such that \( H_1(G - v; \mathbb{Z}) = 0 \). □

![K4 and D3](image)

Fig. 1.4.

Therefore Theorem 1.1 is an unknotting theorem for edge-homotopy on spatial graphs. Our purpose in this paper is to determine when the self delta (resp. sharp) move is a uniforming operation, namely giving an unknotting theorem for delta (resp. sharp) edge-homotopy on spatial graphs. The following is our main result.

**Theorem 1.2.** For a graph \( G \), the following are equivalent.

1. Any two spatial embeddings of \( G \) are delta edge-homotopic.
2. Any two spatial embeddings of \( G \) are sharp edge-homotopic.
3. \( G \) does not contain a subgraph which is homeomorphic to the graph \( \Theta \) as illustrated in Fig. 1.5 or the disjoint union of two 1-spheres.
4. \( G \) is a bouquet, namely there exists a positive integer \( m \) such that \( G \) is homeomorphic to the graph \( B_m \) as illustrated in Fig. 1.5.

We remark here that all of the spatial embeddings of \( \Theta \) and all of the spatial embeddings of the disjoint union of two 1-spheres, namely all *spatial*...
theta curves and all 2-component links, have been classified completely up to delta edge-homotopy [16], [13], and up to sharp edge-homotopy [17], [21]. In the next section we prove lemmas needed later. We prove Theorem 1.2 in section 3.

2. $C_k$-moves on spatial graphs

In this section, we prove lemmas concerning specific local moves needed later. A $C_1$-move is a crossing change and a $C_k$-move ($k \geq 2$) is a local move on a spatial graph as illustrated in Fig. 2.1. This move was introduced by Habiro as a local move on an oriented link [4], [2], and it was extended to spatial graphs by Taniyama and Yasuhara from a standpoint of the “band description” [26] (see also [25], [19]). We note that the original definition of the $C_k$-move is different from the one above, but it is known that each of the original $C_k$-moves can be realized by local moves as illustrated in Fig. 2.1 and ambient isotopies [4]. We note that a $C_2$-move is equivalent to a delta move and a $C_3$-move is called a clasp-pass move [3].

For a $C_k$-move and a self delta move, we have the following.

**Lemma 2.1.** A $C_k$-move is realized by self delta moves and ambient isotopies if at least three of the $(k+1)$ strings in it belong to the same spatial
The statement above was pointed out in [12, p. 179] and the case of \( k = 4 \) was applied to classify 2-component links and spatial theta curves up to delta edge-homotopy effectively [13], [16]. Actually we can show Lemma 2.1 by applying ambient isotopic transformations and \( C_2 \)-moves on the same spatial edge to Fig. 2.1 directly. For example, see Fig. 2.2, where gray parts belong to the same spatial edge. We omit the details.

![Fig. 2.2.](image)

We note that a \( C_k \)-move can be realized by a band sum of a \((k + 1)\)-
A $(k+1)$-component Milnor link is one of the $C_{k-1}$-links, and it is known the following. We refer the reader to [25], [19], [26] for details.

**Lemma 2.2.** Each of the local moves as illustrated in Fig. 2.4 (1), (2) and (3) is realized by $C_k$-moves and ambient isotopies, where $M_k$ denotes a $k$-component Milnor link. □

![Diagram](image)

Therefore, a fusion band with a $k$-component Milnor link can leap over a spatial edge, and a root of a fusion band with a $k$-component Milnor link can pass through a root of a fusion band with another $k$-component Milnor link.

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2. A 2-component Milnor link is a Hopf link, and an $n(\geq 3)$-component Milnor link can be defined as one of the *iterated Bing doubles* of a Hopf link [1].
Fig. 2.4.

link by $C_k$-moves and ambient isotopies. We note that any of the full twists of a fusion band with a $k$-component Milnor link can be cancelled out by Fig. 2.4 (2).

A $C_k$-move is called an adjacent $C_k$-move if all $(k + 1)$ strings in the move belong to exactly $(k + 1)$ mutually adjacent spatial edges. Here we regard a loop as two mutually adjacent edges. The following lemma is a generalization of the facts that a crossing change between adjacent spatial edges is realized by delta moves and ambient isotopies [9] and a delta move between exactly three mutually adjacent spatial edges is realized by clasp-pass moves and ambient isotopies [24, Lemma 3.2].

**Lemma 2.3.** An adjacent $C_k$-move is realized by $C_{k+1}$-moves and ambient isotopies.

**Proof.** We note that any adjacent $C_k$-move can be realized by a band sum of a $(k + 1)$-component Milnor link, where the roots of all of the fusion bands with the link belong to exactly $(k + 1)$ mutually adjacent spatial edges. It is sufficient to show that this Milnor link can be cancelled out by $C_{k+1}$-moves and ambient isotopies. By Lemma 2.2, we can draw the link up sufficiently near by the shared vertex and deform it into the one on the left-hand side in Fig. 2.5 if $k = 1$ and Fig. 2.6 if $k \geq 2$ identically by $C_{k+1}$-moves and ambient isotopies, where $\bigcirc = \bigodot$ or $\bigotimes$. In the case of $k = 1$, this Hopf link is cancelled out up to ambient isotopy. Thus the result holds for $k = 1$. 

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Next we show the case of $k \geq 2$. Let us consider the diagrams as illustrated in Fig. 2.7 which satisfies the following:

1. $c_{i,1} = \begin{array}{l} \circ \quad \text{or} \\ \circ \circ \end{array} (i \neq 1)$.

2. If $c_{0,1} = \begin{array}{l} \circ \quad \text{or} \\ \circ \circ \end{array}$, then $c_{1,1} = \begin{array}{l} \circ \quad \text{or} \\ \circ \circ \end{array}$.

3. If $c_{0,1} = \begin{array}{l} \circ \quad \text{or} \\ \circ \circ \end{array}$, then $c_{1,1} = \begin{array}{l} \circ \quad \text{or} \\ \circ \circ \end{array}$.

4. If $c_{i,1} = \begin{array}{l} \circ \quad \text{or} \\ \circ \circ \end{array}$, then $c_{i,2} = \begin{array}{l} \circ \quad \text{or} \\ \circ \circ \end{array}$.

5. If $c_{i,1} = \begin{array}{l} \circ \quad \text{or} \\ \circ \circ \end{array}$, then $c_{i,2} = \begin{array}{l} \circ \quad \text{or} \\ \circ \circ \end{array}$.

It is easy to see that each of the two diagrams on the lower part is ambient isotopic to the diagram on the upper part. On the other hand, the two diagrams on the lower part are transformed into each other by an obvious adjacent $C_k$-move and ambient isotopies, namely this $C_k$-move is realized by a band sum of a $(k + 1)$-component Milnor link. Then we can produce a $(k + 1)$-component Milnor link with the arbitrary pattern of the half twists of the fusion bands with the link up to $C_{k+1}$-moves and ambient isotopies, see the case of $k = 5$ as illustrated in Fig. 2.8. This implies that an adjacent $C_k$-move can be realized by $C_{k+1}$-moves and ambient isotopies. □
3. Proof of Theorem 1.2

Proof of Theorem 1.2. (1) ⇒ (2): It is known that a delta move is realized by sharp moves on the strings in the delta move and ambient isotopies [14]. Thus the result is clear.

(2) ⇒ (3): We note that if two spatial embeddings $f$ and $g$ of $G$ are sharp edge-homotopic then $f|_H$ and $g|_H$ are sharp edge-homotopic for any subgraph $H$ of $G$. It is known that a spatial theta curve as illustrated in Fig. 3.1 is non-trivial up to sharp edge-homotopy [17, Example 3.6]. Besides we have that a Hopf link is non-trivial up to sharp edge-homotopy because the linking number is a sharp-edge homotopy invariant of an oriented link. So we have the result.

(3) ⇒ (4): Let $G$ be a graph which does not contain a subgraph which is homeomorphic to $\Theta$ or the disjoint union of two 1-spheres. Let $K_5$ and $K_{3,3}$ be the graphs as illustrated in Fig. 3.2. It is well known that a graph is planar if and only if the graph does not contain a subgraph which is homeomorphic to $K_5$ or $K_{3,3}$ [5]. Since each of $K_5$ and $K_{3,3}$ contains a subgraph which is homeomorphic to $\Theta$, we have that $G$ is a planar graph which does not contain mutually disjoint cycles. Thus by [23, Theorem C] we have that $G$ is homeomorphic to a double trident, a multi-spoke wheel or a generalized bouquet. Here a double trident and a multi-spoke wheel are graphs as illustrated in Fig. 3.3 (1) and (2), respectively, where a gray edge is allowed to have arbitrary multiplicity. Since each of double tridents and multi-spoke wheels contains a subgraph which is homeomorphic to $\Theta$, the
Fig. 2.8.
Fig. 3.1.

graph $G$ must be a generalized bouquet. Moreover $G$ must be a bouquet because $G$ does not contain a subgraph which is homeomorphic to $\Theta$. So we have the result.

Fig. 3.2.

(4) $\Rightarrow$ (1): We show that any spatial embedding $f$ of $B_m$ is trivial up to delta edge-homotopy. Let $h$ be the trivial spatial embedding of $B_m$. It is clear that $f$ and $h$ are transformed into each other by $C_1$-moves and ambient isotopies. Then we can regard each of the $C_1$-moves as an adjacent $C_1$-move. Thus by Lemma 2.3 we have that $f$ and $h$ are transformed into each other by $C_2$-moves and ambient isotopies. For each of these $C_2$-moves, if all of the three strings in the $C_2$-move belong to the same knot in $f(B_m)$, it is realized by self delta moves and ambient isotopies by Lemma 2.1.
Otherwise we can regard this $C_2$-move as an adjacent $C_2$-move. Thus by Lemma 2.3 we have that $f$ and $h$ are transformed into each other by $C_3$-moves, self delta moves and ambient isotopies. For each of these $C_3$-moves, if at least three of the four strings in the $C_3$-move belong to the same knot in $f(B_m)$, it is realized by self delta moves and ambient isotopies by Lemma 2.1. Otherwise we can regard this $C_3$-move as an adjacent $C_3$-move. Thus by Lemma 2.3 we have that $f$ and $h$ are transformed into each other by $C_4$-moves, self delta moves and ambient isotopies.

By following the procedure above repeatedly, we have that $f$ and $h$ are transformed into each other by $C_{2m}$-moves, self delta moves and ambient isotopies. Then we can see that for each of the $C_{2m}$-moves there exists a knot in $f(B_m)$ such that at least three of the $(2m + 1)$ strings in the $C_{2m}$-move belong to it. So it is realized by self delta moves and ambient isotopies by Lemma 2.1. Therefore we have that $f$ and $h$ are transformed into each other by self delta moves and ambient isotopies, namely $f$ is delta edge-homotopic to $h$. This completes the proof.

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