An algorithm of computing b-functions

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To Professor Hikosaburo Komatsu on the occasion of his sixtieth birthday

Abstract

We present an algorithm for computing b-functions (Bernstein-Sato polynomials) by using Gröbner bases for rings of differential operators. This algorithm computes the generalized b-function (or the indicial equation) of a holonomic system in general, which enables us to compute, as a special case, the Bernstein-Sato polynomial associated with an arbitrary polynomial.

1 Introduction

Let $f(x) \in K[x] = K[x_1, \dots, x_n]$ be a polynomial of n variables with coefficients in a field K of characteristic zero. Let us denote by

$$A_n(K) := K[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle, \qquad \hat{\mathcal{D}}_n(K) := K[[x_1, \dots, x_n]]\langle \partial_1, \dots, \partial_n \rangle$$

the rings of differential operators with polynomial and formal power series coefficients respectively with $\partial_i = \partial/\partial x_i$ and $\partial = (\partial_1, \dots, \partial_n)$. ($A_n(K)$ is called the Weyl algebra over K.)

Let s be a parameter. Then the (local) b-function (or the Bernstein-Sato polynomial) $b_f(s)$ associated with $f(x)$ is the monic polynomial of the least degree $b(s) \in K[s]$ satisfying

$$P(s, x, \partial) f(x)^{s+1} = b(s) f(x)^s \tag{1.1}$$

with some $P(s, x, \partial) \in \hat{\mathcal{D}}_n(K)[s]$. The monic polynomial of the least degree $b(s) \in K[s]$ satisfying (1.1) with some $P(s, x, \partial) \in A_n(K)[s]$ is denoted by $\tilde{b}_f(s)$. The existence of $\tilde{b}_f(s)$ was proved by I. N. Bernstein [Be1], [Be2], which implies the existence of $b_f(s)$. Note that $b_f(s)$ divides $\tilde{b}_f(s)$ but $b_f(s)$ and $\tilde{b}_f(s)$ are not necessarily identical. More generally, the existence of $b_f(s)$ for $f(x) \in K[[x]]$ was proved by J. E. Björk [Bj].

In this paper, we present an algorithm for, given $f(x) \in K[x]$, computing $b_f(s)$ and finding a $P(s, x, \partial) \in \hat{\mathcal{D}}_n(K)$ satisfying (1.1) with $b(s) = b_f(s)$. More precisely, our algorithm finds a $Q(s, x, \partial) \in A_n(K)[s]$ and an $a(x) \in K[x]$ with $a(0) \neq 0$ such that $P(s, x, \partial) = (1/a(x))Q(s, x, \partial)$ satisfies (1.1) with $b(s) = b_f(s)$. Computing $\tilde{b}_f(s)$ and an associated $P \in A_n(K)[s]$ is slightly easier.

An algorithm of computing $b_f(s)$ was first given by M. Sato et al. [SKKO] when $f(x)$ is a relative invariant of a prehomogeneous vector space. J. Briançon, Ph. Maisonobe et al. [BGMM], [Mai] have given an algorithm of computing $b_f(s)$ for $f(x)$ with isolated singularity. Also note that T. Yano [Y] worked out many interesting examples of b-functions systematically.

Our method consists in computing the (generalized) b-function for a section of a holonomic system (or more generally, a specializable D-module) via Gröbner basis computation in the Weyl algebra. In general, let M be a finitely generated left $A_{n+1}(K)$-module. We write $t = x_{n+1}$, $\partial_t = \partial_{n+1}$, and $\partial = (\partial_1, \ldots, \partial_n)$. Put $\mathcal{M} = \hat{\mathcal{D}}_{n+1}(K) \otimes_{A_{n+1}(K)} M$. For $u \in M$, the (local) b-function of u is the monic polynomial $b_u(s) \in K[s]$ of the least degree, if any, satisfying

$$(b(t\partial_t) + tP(t, x, t\partial_t, \partial))(1 \otimes u) = 0 \quad \text{in } \mathcal{M}$$

with some $P(t, x, t\partial_t, \partial) \in \hat{\mathcal{D}}_{n+1}(K)$ (cf. [KK], [L]). We give an algorithm which computes $b_u(s)$ and P, or determines that there is none, by using a kind of Gröbner basis for left ideals of the Weyl algebra related to a filtration introduced by M. Kashiwara [K3]. Such Gröbner bases were used by Oaku [O2], [O3]. Especially we use the FW-Gröbner basis introduced in [O2]. The computation of the Bernstein-Sato polynomial $b_f(s)$ is reduced to finding the b-function $b_u(s)$ for $u = \delta(t - f(x))$, the delta function supported by $t - f(x) = 0$, as was observed by B. Malgrange [M1] (cf. also [M2], [K1], [K2]).

Our algorithm is strict (at least if K is algebraic over $\mathbf{Q}$) in the sense that, given a finite set of data, it returns an answer as a finite set of data, or else determines that there is no answer, in a finite number of steps using a finite amount of memory in the computation. For example, a system Kan of N. Takayama ([T3]) is available for actual execution of our algorithm.

We could also write down an algorithmic procedure for computing $b_f(s)$ even if $f(x) \in K[[x]]$ by using the FD-Gröbner basis of [O3] instead of the FW-Gröbner basis. However, this would not yield an algorithm in the strict sense as above in general.

2 Gröbner bases for ideals of the Weyl algebra

In this section we recall briefly the theory and algorithm of Gröbner bases for ideals of $A_n(K)$. We fix a (total) order $\prec$ of $\mathbf{N}^{2n}$ with $\mathbf{N} := \{0, 1, 2, \ldots\}$ that satisfies the following conditions:

(A-1) $\alpha \succ \beta$ implies $\alpha + \gamma \succ \beta + \gamma$ for any $\alpha, \beta, \gamma \in \mathbf{N}^{2n}$;

(A-2) $\alpha \succeq 0$ for any $\alpha \in \mathbf{N}^{2n}$.

Under the condition (A-1), the condition (A-2) is equivalent to the order $\prec$ being a well-order (cf. [CLO]).

Let us write $A_n = A_n(K)$ with a field K of characteristic zero. An element P of A_n is written uniquely as a finite sum

$$P = \sum_{\alpha, \beta \in \mathbf{N}^n} a_{\alpha\beta} x^\alpha \partial^\beta$$

with $a_{\alpha\beta} \in K$, where we use the notation $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $\partial^\beta = \partial_1{}^{\beta_1} \dots \partial_n{}^{\beta_n}$ for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$. Then we define the *leading exponent* $\mathrm{lexp}(P)$ and the *leading coeffcient* $\mathrm{lcoef}(P)$ of P by

$$\begin{aligned} \mathrm{lexp}(P) &:= \max{}_\prec \{(\alpha, \beta) \in \mathbf{N}^{2n} \mid a_{\alpha\beta} \neq 0\}, \\ \mathrm{lcoef}(P) &:= a_{\alpha\beta} \quad \text{with } (\alpha, \beta) = \mathrm{lexp}(P), \end{aligned}$$

where $\max_\prec$ denotes taking the maximum element with respect to the order $\prec$ (we assume $P \neq 0$). Let I be a left ideal of A_n. Then the set $E(I)$ of leading exponents of I is defined by

$$E(I) := \{\mathrm{lexp}(P) \mid P \in I \setminus \{0\}\} \subset \mathbf{N}^{2n}.$$

Definition 2.1 (Gröbner basis) A finite subset $\mathbf{G}$ of a left ideal I of A_n is called a *Gröbner basis* of I (with respect to the order $\prec$) if

$$E(I) = \bigcup_{P \in \mathbf{G}} (\mathrm{lexp}(P) + \mathbf{N}^{2n})$$

holds. (Then $\mathbf{G}$ generates I.)

The algorithm of computing a Gröbner basis (i.e. the Buchberger algorithm [Bu]) consists of computing division (or reduction) and S-polynomials (or operators).

Lemma 2.2 (division) *For $P, P_1, \dots, P_k \in A_n$, there exist $Q_1, \dots, Q_k, R \in A_n$ such that*

$$P = \sum_{i=1}^k Q_i P_i + R, \qquad \mathrm{lexp}(R) \notin \bigcup_{i=1}^k (\mathrm{lexp}(P_i) + \mathbf{N}^{2n})$$

and that, for each i, $\mathrm{lexp}(Q_i P_i) \preceq \mathrm{lexp}(P)$ if $Q_i \neq 0$. Moreover, there is an algorithm (the division, or the reduction algorithm) of computing $Q_1, \dots, Q_k, R$.

In general, for vectors $\alpha = (\alpha_1, \dots, \alpha_m)$ and $\beta = (\beta_1, \dots, \beta_m)$ in $\mathbf{N}^m$, we put

$$\alpha \vee \beta := (\max\{\alpha_1, \beta_1\}, \dots, \max\{\alpha_m, \beta_m\}).$$

Definition 2.3 (S-operator) For $P, Q \in A_n$, put $\mathrm{lexp}(P) = (\alpha, \beta)$ and $\mathrm{lexp}(Q) = (\alpha', \beta')$. Then the *S-operator* of P and Q is defined by

$$\mathrm{sp}(P, Q) := \mathrm{lcoef}(Q) x^{\beta \vee \beta' - \beta} \partial^{\alpha \vee \alpha' - \alpha} P - \mathrm{lcoef}(P) x^{\beta \vee \beta' - \beta'} \partial^{\alpha \vee \alpha' - \alpha'} Q.$$

Theorem 2.4 ([G], [C], [T1], [KW]) *Let $\mathbf{G} = \{P_1, \dots, P_k\}$ be a finite subset of A_n which generates a left ideal I of A_n. Then the following two conditions are equivalent:*

(1) $\mathbf{G}$ is a Gröbner basis of I;

(2) For any $i, j \in \{1, \dots, s\}$ with $i < j$, there exist $Q_{ij1}, \dots, Q_{ijk} \in A_n$ so that

$$\mathrm{sp}(P_i, P_j) = \sum_{\ell=1}^k Q_{ij\ell} P_\ell$$

and that $Q_{ij\ell} = 0$ or $\mathrm{lexp}(Q_{ij\ell} P_\ell) \prec \mathrm{lexp}(P_i) \vee \mathrm{lexp}(P_j)$ for each ℓ.

The condition (2) of this theorem provides the Buchberger algorithm of computing a Gröbner basis from a given set of generators.

Proposition 2.5 *Let* $\{P_1, \dots, P_k\}$ *be a Gröbner basis of a left ideal* I *of* A_{n+1}. *Then for* $P \in A_n$, P *belongs to* I *if and only if there exist* $Q_1, \dots, Q_k \in A_n$ *such that* $P = \sum_{i=1}^k Q_i P_i$ *and that* $\mathrm{lexp}(Q_i P_i) \preceq \mathrm{lexp}(P)$ *if* $Q_i \neq 0$ *for each* i.

The following theorem can be proved in the same way as its counterpart in the polynomial ring ([CLO],[BW],[E]):

Theorem 2.6 *Let* $\mathbf{G} = \{P_1, \dots, P_k\}$ *be a Gröbner basis of a left ideal* I *of* A_n. *Take* $Q_{ij\ell}$ *satisfying the condition (2) of Theorem 2.4. Put* $\mathrm{lexp}(P_i) = (\alpha^{(i)}, \beta^{(i)})$ *and*

$$\begin{aligned} S_{ji} &:= \mathrm{lcoef}(P_i) x^{\beta^{(i)} \vee \beta^{(j)} - \beta^{(i)}} \partial^{\alpha^{(i)} \vee \alpha^{(j)} - \alpha^{(i)}}, \\ \vec{V}_{ij} &:= (0, \dots, \overset{(i)}{S_{ji}}, \dots, -\overset{(j)}{S_{ij}}, \dots, 0) - (Q_{ij1}, \dots, Q_{ijk}) \ \in (A_n)^k. \end{aligned}$$

Then the syzygy module

$$S(P_1, \dots, P_k) := \{(Q_1, \dots, Q_k) \in (A_n)^k \mid \sum_{j=1}^k Q_j P_j = 0\}$$

is generated by $\{\vec{V}_{ij} \mid 1 \leq i < j \leq k\}$.

3 Gröbner basis and homogenization with respect to a filtration

Fixing a field K of characteristic zero, we put

$$\begin{aligned} A_{n+1} = A_{n+1}(K) &:= K[t, x_1, \dots, x_n]\langle \partial_t, \partial_1, \dots, \partial_n \rangle, \\ \hat{\mathcal{D}}_{n+1} = \hat{\mathcal{D}}_{n+1}(K) &:= K[[t, x_1, \dots, x_n]]\langle \partial_t, \partial_1, \dots, \partial_n \rangle \end{aligned}$$

with $\partial_t := \partial/\partial t$ and $\partial_i = \partial/\partial x_i$. Put $Y = \{(t, x) \in K^{n+1} \mid t = 0\}$. We use a filtration ($V$-filtration) with respect to Y introduced by Kashiwara [K3] for the study of vanishing cycle sheaves (cf. also [L], [LS]). An element P of A_{n+1} (or of $\hat{\mathcal{D}}_{n+1}$) is written in the form

$$P = \sum_{\mu, \nu \geq 0, \alpha, \beta \in \mathbf{N}^n} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha {\partial_t}^\nu \partial^\beta. \tag{3.1}$$

For each integer m, define K-subspaces of A_{n+1} and of $\hat{\mathcal{D}}_{n+1}$ respectively by

$$\begin{aligned} F_m(A_{n+1}) &:= \{P = \sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha \partial_t^\nu \partial^\beta \in A_{n+1} \mid a_{\mu,\nu,\alpha,\beta} = 0 \text{ if } \nu - \mu > m\}, \\ F_m(\hat{\mathcal{D}}_{n+1}) &:= \{P = \sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha \partial_t^\nu \partial^\beta \in \hat{\mathcal{D}}_{n+1} \mid a_{\mu,\nu,\alpha,\beta} = 0 \text{ if } \nu - \mu > m\}. \end{aligned}$$

For a nonzero element P of $\hat{\mathcal{D}}_{n+1}$, we define the *F-order* $\mathrm{ord}_F(P)$ of P as the minimum integer m that satisfies $P \in F_m(\hat{\mathcal{D}}_{n+1})$. When the F-order of P in the form (3.1) is m, we put

$$\hat{\sigma}(P) = \hat{\sigma}_m(P) := \sum_{\nu-\mu=m} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha \partial_t^\nu \partial^\beta$$

and call it the *formal symbol* of P along Y (cf. [LS]). We have $\mathrm{ord}_F(PQ) = \mathrm{ord}_F(P) + \mathrm{ord}_F(Q)$ and $\hat{\sigma}(PQ) = \hat{\sigma}(P)\hat{\sigma}(Q)$ for $P, Q \in \hat{\mathcal{D}}_{n+1}$.

Now let $\prec_F$ be an order on $\mathbf{N}^{2n+2}$ which satisfies (A-1) (with n replaced by $n+1$) and

(A-3) if $\nu - \mu > \nu' - \mu'$, then $(\mu, \nu, \alpha, \beta) \succ_F (\mu', \nu', \alpha', \beta')$;

(A-4) $(\mu, \mu, \alpha, \beta) \succeq_F (0, 0, 0, 0)$ for any $\mu \in \mathbf{N}$ and $\alpha, \beta \in \mathbf{N}^n$.

The condition (A-3) implies that $\prec_F$ is not a well-order. However, the definitions in the preceding section apply to this order $\prec_F$. The only difficulty is that Lemma 2.2 does not hold in general. Let us denote by $\mathrm{lexp}_F(P) \in \mathbf{N}^{2n+2}$, $\mathrm{lcoef}_F(P) \in K$ the leading exponent and the leading coefficient of $P \in A_{n+1} \setminus \{0\}$ with respect to $\prec_F$ respectively. The set of leading exponents $E_F(I) \subset \mathbf{N}^{2n+2}$ is defined in the same way.

Definition 3.1 (FW-Gröbner basis) A finite set $\mathbf{G}$ of generators of a left ideal I of A_{n+1} is called an FW-*Gröbner basis* of I if we have

$$E(I) = \bigcup_{P \in \mathbf{G}} (\mathrm{lexp}(P) + \mathbf{N}^{2n+2}).$$

Since we do not have division algorithm, the Buchberger algorithm does not work directly. In order to bypass this difficulty to obtain an algorithm of computing FW-Gröbner bases, we use the homogenization technique.

Definition 3.2 For $i, j, \mu, \nu, \mu', \nu' \in \mathbf{N}$ and $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$, an order $\prec_H$ on $\mathbf{N}^{2n+3}$ is defined by

$$\begin{aligned}(i, \mu, \nu, \alpha, \beta) \succ_H (j, \mu', \nu', \alpha', \beta') \quad \text{if and only if} \quad & (i > j) \quad \text{or} \quad (i = j \quad \text{and} \\ & (\mu + \ell, \nu, \alpha, \beta) \succ_F (\mu' + \ell', \nu', \alpha', \beta')) \\ & \text{or } (i = j,\ \nu = \nu',\ \alpha = \alpha',\ \beta = \beta', \mu > \mu')\end{aligned}$$

with $\ell, \ell' \in \mathbf{N}$ such that $\nu - \mu - \ell = \nu' - \mu' - \ell'$. This definition is independent of the choice of ℓ, ℓ' in view of the condition (A-1).

Lemma 3.3 *(1) $\prec_H$ is a well-order.*

(2) If $\nu - \mu - i = \nu' - \mu' - j$, then $(i, \mu, \nu, \alpha, \beta) \succ_H (j, \mu', \nu', \alpha', \beta')$ if and only if $(\mu, \nu, \alpha, \beta) \succ_F (\mu', \nu', \alpha', \beta')$.

Proof: (1) It suffices to show $(i,\mu,\nu,\alpha,\beta) \succeq_H (0,0,0,0,0)$ for any $i,\mu,\nu \in \mathbf{N}$ and $\alpha,\beta \in \mathbf{N}^n$. Since this is obvious when $i > 0$, let us assume $i = 0$. Take $\ell, \ell' \in \mathbf{N}$ so that $\nu - \mu - \ell = -\ell'$. Then we have

$$(\mu + \ell - \ell', \nu, \alpha, \beta) = (\nu, \nu, \alpha, \beta) \succeq_F (0,0,0,0)$$

by (A-4). This implies $(\mu + \ell, \nu, \alpha, \beta) \succeq_F (\ell', 0, 0, 0)$ in view of (A-1). Hence we have $(0,\mu,\nu,\alpha,\beta) \succeq_H (0,0,0,0,0)$ by definition.

(2) Assume $\nu - \mu - i = \nu' - \mu' - j$. Then we have $\nu - \mu > \nu' - \mu'$ if and only if $i > j$. If $i = j$, we can take $\ell = \ell' = 0$ in Definition 3.2. The assertion (2) follows from these facts combined with (A-3) and the definition of $\prec_H$. This completes the proof. □

For a nonzero element $P = P(x_0)$ of $A_{n+1}[x_0]$, let us denote by $\mathrm{lexp}_H(P) \in \mathbf{N}^{2n+3}$ and $\mathrm{lcoef}_H(P) \in K$ the leading exponent and the leading coefficient of P with respect to $\prec_H$ respectively. The set $E_H(I)$ of leading exponents of a left ideal I of $A_{n+1}[x_0]$ is also defined.

Definition 3.4 (F-homogeneity) An element P of $A_{n+1}[x_0]$ of the form

$$P = \sum_{i,\mu,\nu,\alpha,\beta} a_{i,\mu,\nu,\alpha,\beta} {x_0}^i t^\mu x^\alpha \partial_t^\nu \partial^\beta$$

is said to be *F-homogeneous* of order m if $a_{i,\mu,\nu,\alpha,\beta} = 0$ whenever $\nu - \mu - i \neq m$.

Definition 3.5 (F-homogenization) For an element P of A_{n+1} of the form

$$P = \sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta} t^\mu x^\alpha \partial_t^\nu \partial^\beta,$$

put $m = \min\{\nu - \mu \mid a_{\mu,\nu,\alpha,\beta} \neq 0 \text{ for some } \alpha,\beta \in \mathbf{N}^n\}$. Then the *F-homogenization* $P^h \in A_{n+1}[x_0]$ of P is defined by

$$P^h = \sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta} {x_0}^{\nu-\mu-m} t^\mu x^\alpha \partial_t^\nu \partial^\beta.$$

P^h is F-homogeneous of order m.

The following two lemmas follow from the Leibniz rule for the product of differential operators.

Lemma 3.6 *If $P, Q \in A_{n+1}[x_0]$ are both F-homogeneous, then so is PQ.*

Lemma 3.7 *For $P, Q \in A_{n+1}$, we have $(PQ)^h = P^h Q^h$.*

The following lemma is an immediate consequence of the definition.

Lemma 3.8 *For $P_1, \dots, P_k \in A_{n+1}$, put $P = P_1 + \dots + P_k$. Then there exist $\ell, \ell_1, \dots, \ell_k \in$ $\mathbf{N}$ so that*

$${x_0}^\ell P^h = {x_0}^{\ell_1} (P_1)^h + \dots + {x_0}^{\ell_k} (P_k)^h.$$

Let us define $\varpi : \mathbf{N}^{2n+3} \longrightarrow \mathbf{N}^{2n+2}$ by $\varpi(i, \mu, \nu, \alpha, \beta) = (\mu, \nu, \alpha, \beta)$.

Lemma 3.9 *(1) If $P(x_0) \in A_{n+1}[x_0]$ is F-homogeneous, then we have $\mathrm{lexp}_F(P(1)) = \varpi(\mathrm{lexp}_H(P(x_0)))$.*

(2) For any A_{n+1}, we have $\mathrm{lexp}_F(P) = \varpi(\mathrm{lexp}_H(P^h))$.

Proof: (1) follows from (2) of Lemma 3.3. (2) follows from (1) since $P^h(1) = P$. □

Proposition 3.10 *Let I be a left ideal of $A_{n+1}[x_0]$ generated by F-homogeneous operators. Then there exists an H-Gröbner basis (i.e. a Gröbner basis with respect to $\prec_H$) of I consisting of F-homogeneous operators. Moreover, such an H-Gröbner basis can be computed by the Buchberger algorithm.*

Proof: Since $\prec_H$ is a well-order, the arguments in Section 2 apply. Hence we have only to verify that taking the S-operator and computing division both preserve the F-homogeneity. This follows from Lemma 3.6. □

Proposition 3.11 *Let I be a left ideal of A_{n+1} generated by $P_1, \ldots, P_d \in A_{n+1}$. Let us denote by I^h the left ideal of $A_{n+1}[x_0]$ generated by $(P_1)^h, \ldots, (P_d)^h$. (Here I^h is not defined uniquely by I.) Let $\mathbf{G} = \{Q_1(x_0), \ldots, Q_k(x_0)\}$ be an H-Gröbner basis of I^h consisting of F-homogeneous operators. Then for $P \in A_{n+1}$ the following two conditions are equivalent:*

(1) $P \in I$;

(2) there exist $U_1, \ldots, U_k \in A_{n+1}$ such that $P = \sum_{j=1}^{k} U_j Q_j(1)$, and that for each $j = 1, \ldots, k$, $\mathrm{lexp}_F(U_j Q_j(1)) \preceq_F \mathrm{lexp}_F(P)$ if $U_j \neq 0$.

Proof: Assume $P \in I$. Then there exist $V_1, \ldots, V_d \in A_{n+1}$ such that $P = V_1 P_1 + \ldots + V_d P_d$. Then by Lemmas 3.7 and 3.8, there exist $\ell, \ell_1, \ldots, \ell_d \in \mathbf{N}$ such that

$$x_0{}^{\ell} P^h(x_0) = x_0{}^{\ell_1} (V_1)^h(x_0)(P_1)^h(x_0) + \ldots + x_0{}^{\ell_d} (V_d)^h(x_0)(P_d)^h(x_0) \ \in I^h.$$

Hence by Proposition 2.5 there exist F-homogeneous $U_1(x_0), \ldots, U_k(x_0) \in A_{n+1}[x_0]$ such that

$$x_0{}^{\ell} P^h(x_0) = U_1(x_0)Q_1(x_0) + \ldots + U_k(x_0)Q_k(x_0)$$

and that $\mathrm{lexp}_H(U_j(x_0)Q_j(x_0)) \preceq_H \mathrm{lexp}_H(x_0{}^{\ell} P^h(x_0))$ for each j with $U_j(x_0) \neq 0$. Setting $x_0 = 1$, we see that (2) holds in view of Lemmas 3.3 and 3.9.

In order to prove the inverse implication, it suffices to show that each $Q_j(1)$ belongs to I. Since $Q_j(x_0) \in I^h$, there exist $V_1(x_0), \ldots, V_d(x_0) \in A_{n+1}[x_0]$ such that

$$Q_j(x_0) = V_1(x_0)(P_1)^h(x_0) + \ldots + V_d(x_0)(P_d)^h(x_0).$$

Hence we get $Q_j(1) \in I$ setting $x_0 = 1$. This completes the proof. □

Theorem 3.12 *$\mathbf{G}(1) := \{Q_1(1), \ldots, Q_k(1)\}$ is an FW-Gröbner basis of I under the same assumptions as in Proposition 3.11.*

Proof: In view of Proposition 3.11, $\mathbf{G}(1)$ generates I. Let P be an arbitrary element of I. Then it suffices to show that

$$\mathrm{lexp}_F(P) \in \bigcup_{j=1}^{k}(\mathrm{lexp}_F(Q_j(1) + \mathbf{N}^{2n+2}).$$

Since $P \in I$, we can take U_j satisfying the condition (2) of Proposition 3.11. Then we have

$$\mathrm{lexp}_F(P) = \mathrm{lexp}_F(U_j Q_j(1)) = \mathrm{lexp}_F(U_j) + \mathrm{lexp}_F(Q_j(1)) \in \mathrm{lexp}_F(Q_j(1)) + \mathbf{N}^{2n+2}$$

for some j. This completes the proof. □

Theorem 3.13 *Under the same assumptions as in Proposition 3.11, there exist, for any $i, j \in \{1, \ldots, k\}$ with $i < j$, F-homogeneous $U_{ij1}(x_0), \ldots, U_{ijk}(x_0) \in A_{n+1}[x_0]$ so that*

$$\mathrm{sp}(Q_i(x_0), Q_j(x_0)) = S_{ji}(x_0)Q_i(x_0) - S_{ij}(x_0)Q_j(x_0) = \sum_{\ell=1}^{k} U_{ij\ell}Q_\ell(x_0), \tag{3.2}$$

where $S_{ji}(x_0)$ is defined in the same way as S_{ji} in Theorem 2.6, and, for each ℓ, we have $\mathrm{lexp}_H(U_{ij\ell}Q_\ell(x_0)) \prec \mathrm{lexp}_F(Q_i(x_0)) \vee \mathrm{lexp}_F(Q_j(x_0))$, if $U_{ij\ell}(x_0) \neq 0$. Put

$$\vec{V}_{ij}(x_0) = (0, \ldots, \overset{(i)}{S_{ji}}(x_0), \ldots, -\overset{(j)}{S_{ij}}(x_0), \ldots, 0) - (U_{ij1}(x_0), \ldots, U_{ijk}(x_0)) \ \in (A_{n+1}[x_0])^k.$$

Then the syzygy module

$$S(Q_1(1), \ldots, Q_k(1)) := \{(U_1, \ldots, U_k) \in (A_{n+1})^k \mid \sum_{\ell=1}^{k} U_\ell Q_\ell(1) = 0\}$$

is generated by $\{\vec{V}_{ij}(1) \mid 1 \leq i < j \leq k\}$.

Proof: The first assertion follows from Theorem 2.4. Suppose

$$(U_1, \ldots, U_k) \in S(Q_1(1), \ldots, Q_k(1)).$$

Then in view of Lemmas 3.7 and 3.8, there exist $\ell_1, \ldots, \ell_k \in \mathbf{N}$ such that

$$x_0{}^{\ell_1}(U_1)^h(x_0)Q_1(x_0) + \ldots + x_0{}^{\ell_k}(U_k)^h(x_0)Q_k(x_0) = 0.$$

Hence by applying Theorem 2.6 to the order $\prec_H$, there are $W_{ij}(x_0) \in A_{n+1}[x_0]$ such that

$$(x_0{}^{\ell_1}(U_1)^h(x_0), \ldots, x_0{}^{\ell_k}(U_k)^h(x_0)) = \sum_{i<j} W_{ij}(x_0)\vec{V}_{ij}(x_0).$$

This completes the proof by setting $x_0 = 1$. □

Definition 3.14 Let P be a nonzero element of A_{n+1} (resp. $\hat{\mathcal{D}}_{n+1}$) of F-order m. Then we define $\psi(P)(s) \in A_n[s]$ (resp. $\hat{\mathcal{D}}_n[s]$) by

$$\hat{\sigma}_0(t^m P) = \psi(P)(t\partial_t) \quad \text{if} \quad m \geq 0,$$
$$\hat{\sigma}_0(\partial_t^{-m} P) = \psi(P)(t\partial_t) \quad \text{if} \quad m < 0.$$

The cause of our use of FW-Gröbner basis lies in the following theorem:

Theorem 3.15 *We use the same notation as in Proposition 3.11. Let $\psi(I)$ be the left ideal of $A_n[s]$ generated by the set $\{\psi(P)(s) \mid P \in I \cap (F_0(A_{n+1}) \setminus F_{-1}(A_{n+1}))\}$. Then $\psi(I)$ is generated by $\psi(Q_1(1)), \ldots, \psi(Q_k(1))$.*

Proof: By definition it is easy to see that $\psi(Q_j(1)) \in \psi(I)$ for $j = 1, \ldots, k$. Suppose $P \in I \cap (F_0(A_{n+1}) \setminus F_{-1}(A_{n+1}))$. Let $U_1, \ldots, U_k \in A_{n+1}$ be as in Proposition 3.11. Let $Q_j(1)$ be of F-order m_j. Then the F-order of U_j is not greater than $-m_j$. Hence we can take $U'_j \in F_0(A_{n+1})$ such that $\hat{\sigma}_{-m_j}(U_j) = U'_j S_j$, where $S_j = t^{m_j}$ if $m_j \geq 0$ and $S_j = \partial_t{}^{-m_j}$ if $m_j < 0$. Then we have

$$\psi(P)(t\partial_t) = \sum_{j=1}^{k} \hat{\sigma}_0(U_j Q_j(1)) = \sum_{j=1}^{k} \hat{\sigma}_{-m_j}(U_j)\hat{\sigma}_{m_j}(Q_j(1)) = \sum_{j=1}^{k} U'_j \psi(Q_j(1))(t\partial_t).$$

This completes the proof. □

What is more crucial in the application of FW-Gröbner bases to the D-module theory is that this theorem can be localized as follows:

Theorem 3.16 *In the same notation as Theorem 3.15, let $\mathcal{I} = \hat{\mathcal{D}}_{n+1} I$ be the left ideal of $\hat{\mathcal{D}}_{n+1}$ generated by I. Let $\psi(\mathcal{I})$ be the left ideal of $\hat{\mathcal{D}}_n[s]$ generated by the set $\{\psi(P)(s) \mid P \in \mathcal{I} \cap (F_0(\hat{\mathcal{D}}_{n+1}) \setminus F_{-1}(\hat{\mathcal{D}}_{n+1}))\}$. Then $\psi(\mathcal{I})$ is generated by $\psi(Q_1(1)), \ldots, \psi(Q_k(1))$.*

Proof: We use the same notation as in Theorem 3.13. In general, for a nonzero element $P = P(x_0)$ of $A_{n+1}[x_0]$, let us denote by $\sigma_H(P)(x_0)$ the highest degree part of P with respect to x_0. Then we have $\hat{\sigma}(P(1)) = \sigma_H(P)(1)$. By taking the maximum degree parts with respect to x_0 in the both sides of the equation (3.2) of Theorem 3.13, we know that $\{\sigma_H(Q_1)(x_0), \ldots, \sigma_H(Q_k)(x_0)\}$ is an H-Gröbner basis of a left ideal of $A_{n+1}[x_0]$.

Let $Q_j(1)$ be of F-order m_j and set $m_{ij} = \nu - \mu$ with $(\mu, \nu, \alpha, \beta) = \mathrm{lexp}_F(Q_i(1)) \vee \mathrm{lexp}_F(Q_j(1))$. Put

$$\vec{V}^0_{ij}(x_0) := (0, \ldots, \sigma_H(\overset{(i)}{S_{ji}})(x_0), \ldots, -\sigma_H(\overset{(j)}{S_{ij}})(x_0), \ldots, 0) - (U^0_{ij1}(x_0), \ldots, U^0_{ijk}(x_0)).$$

Here we put $U^0_{ij\ell}(x_0) := \sigma_H(U_{ij\ell})(x_0)$ if $\mathrm{ord}_F(U_{ij\ell}(1)Q_j(1)) = m_{ij} - m_\ell$, and otherwise $U^0_{ij\ell}(x_0) = 0$. Hence by applying Theorem 3.13 to $\sigma_H(Q_j)(x_0)$, we know that the syzygy module $S(\hat{\sigma}(Q_1(1)), \ldots, \hat{\sigma}(Q_k(1)))$ is generated by $\mathbf{V}^0 := \{\vec{V}^0_{ij}(1) \mid i < j\}$. By virtue of the flatness of $\hat{\mathcal{D}}_{n+1}$ over A_{n+1}, this implies that the syzygy module

$$\hat{S} := \{(U_1, \ldots, U_k) \in (\hat{\mathcal{D}}_{n+1})^k \mid \sum_{j=1}^{k} U_j \hat{\sigma}(Q_j(1)) = 0\}$$

is also generated by $\mathbf{V}^0$ over $\hat{\mathcal{D}}_{n+1}$.

Now suppose $P \in \mathcal{I} \cap (F_0(\hat{\mathcal{D}}_{n+1}) \setminus F_{-1}(\hat{\mathcal{D}}_{n+1}))$. Then there exist $U_1, \dots, U_k \in \hat{\mathcal{D}}_{n+1}$ such that

$$P = U_1 Q_1(1) + \dots + U_k Q_k(1).$$

Put $m := \max\{\mathrm{ord}_F(U_j Q_j(1)) \mid j = 1, \dots, k\}$. Our aim is to show that we can choose $U_1, \dots, U_k$ so that $m = \mathrm{ord}_F(P)$. For this purpose, assume $m > \mathrm{ord}_F(P)$. Put $U_j^0 = \hat{\sigma}_{m-m_j}(U_j)$. Then we have $(U_1^0, \dots, U_k^0) \in \hat{S}$. Hence there exist F-homogeneous $W_{ij} \in \hat{\mathcal{D}}_{n+1}$ such that

$$(U_1^0, \dots, U_k^0) = \sum_{i<j} W_{ij} \vec{V}_{ij}^0(1).$$

Define $U_1', \dots, U_k' \in \hat{\mathcal{D}}_{n+1}$ by

$$(U_1', \dots, U_k') = \sum_{i<j} W_{ij} \vec{V}_{ij}(1).$$

Then we get

$$P = (U_1 - U_1') Q_1(1) + \dots + (U_k - U_k') Q_k(1).$$

We also have $\mathrm{ord}_F((U_j - U_j') Q_j(1)) < m$ since $\hat{\sigma}_{m-m_j}(U_j) = \hat{\sigma}_{m-m_j}(U_j')$. By using this argument repeatedly, we can choose $U_1, \dots, U_k \in \hat{\mathcal{D}}_{n+1}$ such that $P = \sum_{j=1}^k U_j Q_j(1)$ and that $\mathrm{ord}_F(U_j Q_j(1)) \le \mathrm{ord}_F(P)$ for any j. Hence we know, by the same argument as in the proof Theorem 3.15, that $\psi(\mathcal{I})$ is generated by $\psi(Q_1(1)), \dots, \psi(Q_k(1))$. This completes the proof. □

Remark 3.17 Theorems 3.15 and 3.16 hold with $\{Q_1(1), \dots, Q_1(k)\}$ replaced by an arbitrary FW-Gröbner basis of I. To prove this, we have only to develop a theory of FW-Gröbner basis by using the 'truncated division' with respect to the filtration as in [O3].

4 Computation of the b-function of a D-module

We retain the notation in the preceding section. Let M be a finitely generated left A_{n+1}-module and u a nonzero element of M. In the sequel, we assume that a system of the equations for u is given explicitly; i.e., we assume that a finite set of generators of a left ideal I of A_{n+1} is given so that $A_{n+1} u \simeq A_{n+1}/I$.

More generally, if a presentation of M and a representation of u is known, i.e., if generators of a left A_{n+1}-submodule N of $(A_{n+1})^r$ is given so that $M \simeq (A_{n+1})^r/N$, and also given is an element $\vec{U} \in (A_{n+1})^r$ such that u corresponds to the modulo class of $\vec{U}$ by the above isomorphism, then there is an algorithm to find generators of the above I by computing syzygies by means of (a generalization of) Theorem 2.6.

Put $\mathcal{M} := \hat{\mathcal{D}}_{n+1} \otimes_{A_{n+1}} M$ and $\mathcal{I} := \hat{\mathcal{D}}_{n+1} I$. Then we have $\hat{\mathcal{D}}_{n+1}(1 \otimes u) \simeq \hat{\mathcal{D}}_{n+1}/\mathcal{I}$. The *b-function* $b_u(s)$ of u is the monic polynomial, if any, $b(s) \in K[s]$ of the least degree satisfying

$$(b(t\partial_t) + P)(1 \otimes u) = 0 \quad \text{in } \mathcal{M} \qquad (4.1)$$

with some $P \in F_{-1}(\hat{\mathcal{D}}_{n+1})$. Note that (4.1) is equivalent to $b(t\partial_t) + P \in \mathcal{I}$.

The existence of the b-function in this sense was proved by Kashiwara-Kawai [KK], Laurent [L] when $K = \mathbf{C}$ and $\mathcal{M}$ is holonomic. Our purpose here is to present an algorithm to determine whether there exists such nonzero $b(s)$, and if it does, to find $b_u(s)$ and an associated P.

Now let M, u, I be as above and let $\mathbf{G} = \{Q_1(x_0), \ldots, Q_k(x_0)\}$ be an H-Gröbner basis of I^h consisting of F-homogeneous elements as in the preceding section. Let $\psi(I)$ and $\psi(\mathcal{I})$ be left ideals of $A_{n+1}[s]$ and of $\hat{\mathcal{D}}_{n+1}[s]$ respectively defined in Theorems 3.15 and 3.16. Then $\psi(I)$ and $\psi(\mathcal{I})$ are both generated by $\psi(\mathbf{G}(1)) := \{\psi(Q_1(1)), \ldots, \psi(Q_k(1))\}$.

Let $\prec$ be an order on $\mathbf{N}^{2n+1}$ satisfying (A-1), (A-2) (with $2n+2$ replaced by $2n+1$) and

(A-5) if $|\beta| > |\beta'|$, then $(\mu, \alpha, \beta) \succ (\mu', \alpha', \beta')$ for any $\mu, \mu' \in \mathbf{N}$ and $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$.

In particular, if the order $\prec_F$ satisfies, in addition to (A-1), (A-3), (A-4),

(A-6) if $\nu - \mu = \nu' - \mu'$ and $|\beta| > |\beta'|$, then $(\mu, \nu, \alpha, \beta) \succ (\mu', \nu', \alpha', \beta')$ for any $\mu, \nu, \mu', \nu' \in \mathbf{N}$ and $\alpha, \beta, \alpha', \beta' \in \mathbf{N}^n$,

then the order $\prec_{F'}$ on $\mathbf{N}^{2n+1}$ defined by

$$(\mu, \alpha, \beta) \succ_{F'} (\mu', \alpha', \beta') \iff (\mu, \mu, \alpha, \beta) \succ_F (\mu', \mu', \alpha', \beta')$$

satisfies (A-1),(A-2),(A-5).

For an element P of $A_n[s]$ (resp. $\hat{\mathcal{D}}_n[s]$), its order $\mathrm{ord}(P)$ is defined to be the usual order with respect to ∂, and the principal symbol $\sigma(P) \in K[x, \xi, s]$ (resp. $K[[x]][\xi, s]$) is defined also in the standard way with $\xi = (\xi_1, \ldots, \xi_n)$.

Theorem 4.1 *Let $\sigma(\psi(I))$ and $\sigma(\psi(\mathcal{I}))$ be ideals of $K[x, \xi, s]$ and of $K[[x]][\xi, s]$ generated by $\{\sigma(P) \mid P \in \psi(I)\}$ and by $\{\sigma(P) \mid P \in \psi(\mathcal{I})\}$ respectively. Let $\mathbf{H}$ be a Gröbner basis of $\psi(I)$ with respect to an order $\prec$ on $\mathbf{N}^{2n+1}$ satisfying (A-1), (A-2), (A-5). Then $\sigma(\psi(I))$ and $\sigma(\psi(\mathcal{I}))$ are both generated by $\sigma(\mathbf{H}) := \{\sigma(P) \mid P \in \mathbf{H}\}$. In particular, if the order $\prec_F$ satisfies (A-6) in addition to (A-1), (A-3), (A-4), then we can take $\sigma(\psi(\mathbf{G}(1))) := \{\sigma(\psi(Q_1(1))), \ldots, \sigma(\psi(Q_k(1)))\}$ as H.*

Proof: The first assertion can be proved in the same way as Theorem of [O1]; one has only to modify its proof by adding a parameter s. The last statement amounts to saying that $\psi(\mathbf{G}(1))$ is a Gröbner basis with respect to $\prec$. In fact, (A-6) and Proposition 3.11 guarantee that this is the case. $\Box$

Corollary 4.2 *In the same notation as in Theorem 4.1, put $J := \psi(I) \cap K[x, s]$ and $\mathcal{J} := \psi(\mathcal{I}) \cap K[[x]][s]$. Then J and $\mathcal{J}$ are both generated by $\sigma(\mathbf{H}) \cap K[x, s]$ as ideals of $K[x, s]$ and of $K[[x]][s]$ respectively.*

Corollary 4.3 *The b-function of u is the monic generator of $\mathcal{J} \cap K[s]$ if it is not the zero ideal. If $\mathcal{J} \cap K[s] = 0$, then the b-function does not exist.*

At this stage, the problem has become one in commutative algebra. Hence the arguments below should be more or less standard. To be general, let J be an ideal of $K[x,s]$ whose generators are given explicitly and put $\mathcal{J} = K[[x]][s]J$. Our purpose is to compute $\mathcal{J} \cap K[s]$. ($J \cap K[s]$ is computed easily through the elimination by Gröbner basis.)

The following lemma is a consequence of the faithful flatness of $K[[x]]$ over $K[x]_0$:

Lemma 4.4 *Let $K[x]_0$ be the localization of $K[x]$ with respect to the maximal ideal generated by $x_1, \ldots, x_n$. Put $\mathcal{J}' := K[x]_0[s]J$. Then we have $\mathcal{J} \cap K[s] = \mathcal{J}' \cap K[s]$.*

Thus we can compute $\mathcal{J} \cap K[s]$ by using the Gröbner basis computation in the polynomial ring and factorization in $\overline{K}[s]$ in the following steps (we denote by $\overline{K}$ the algebraic closure of K):

Algorithm 4.5 Input: a set of generators $f_1(x,s), \ldots, f_k(x,s)$ of J:

(1) Determine whether there exists, and find if any, some $g(x,s) \in J$ such that its leading coefficient with respect to s does not vanish at $x = 0$; This can be done, e.g., as follows (one can use, instead of the homogenization below, Mora's tangent cone algorithm ([Mo])):

   (a) Let $(f_i)^h(x_0, x, s)$ be the homogenization of $f_i(x,s)$ with respect to x; i.e., $(f_i)^h$ is homogeneous with respect to x_0 and x, and $(f_i)^h(1,x,s) = f_i(x,s)$;

   (b) Let $>$ be an order on $\mathbf{N} \times \mathbf{N} \times \mathbf{N}^n \ni (i, \mu, \alpha)$ with (i, μ, α) corresponding to ${x_0}^i s^\mu x^\alpha$. Assume $>$ satisfies (A-1), (A-2) and $(i,\mu,\alpha) > (j,\nu,\beta)$ if $\mu > \nu$ or ($\mu = \nu$ and $i > j$);

   (c) Let $\{g_1(x_0,x,s), \ldots, g_r(x_0,x,s)\}$ be a Gröbner basis of the ideal generated by $(f_1)^h, \ldots, (f_k)^h$ with respect to $>$.

   (d) Let $g(x,s)$ be one of $g_i(1,x,s)$ with the property above; if there is no such $g(x,s)$, then quit (there is no $b(s)$);

(2) Compute the monic generator $f_0(s)$ of the ideal $J(0)$ of $K[s]$ that is generated by $f_1(0,s), \ldots, f_k(0,s)$ by Gröbner basis or GCD computation; if $f_0(s) = 1$, then put $b(s) := 1$ and go to (6);

(3) Compute the factorization $f_0(s) = (s - s_1)^{\mu_1} \ldots (s - s_m)^{\mu_m}$ in $\overline{K}[s]$;

(4) Put $\overline{J} := \overline{K}[x,s]J$. For each $i = 1, \ldots, m$, determine the least integer $\ell_i = \ell \geq 0$ satisfying $h(x,s)(s - s_i)^\ell \in J$ with some $h(x,s) \in \overline{K}[x,s]$ such that $h(0,s_i) \neq 0$, or else determine that there is no such ℓ; This can be done by computing ideal quotient and saturation via Gröbner bases as follows (cf. [BW], [CLO], [E]):

   For $i := 1$ to m do

   (a) Compute a set of generators G_i of the saturation $\overline{J} : (s - s_i)^\infty$ by means of Gröbner basis;

   (b) Determine whether there is some $h(x,s) \in G_i$ such that $h(0,s_i) \neq 0$; if there is no such h, then put $\ell_i := \infty$ and quit (there is no $b(s)$);

(c) By computing the ideal quotient $\overline{J} : (s - s_i)^\ell$ for $\ell = \mu_i, \mu_i + 1, \dots$ repeatedly, determine the least $\ell \geq \mu_i$ such that $\overline{J} : (s - s_i)^\ell$ contains an element which does not vanish at $(x, s) = (0, s_i)$. Denote this ℓ by ℓ_i;

(5) Put $b(s) := (s - s_1)^{\ell_1} \dots (s - s_m)^{\ell_m}$; we have $b(s) \in K[s]$;

(6) Compute $J_0 := (J : b(s)) \cap K[x]$ by Gröbner bases and find an element $a(x) \in J_0$ such that $a(0) \neq 0$; Such $a(x)$ exists;

(7) Find by division $q_1(x, s), \dots, q_k(x, s) \in K[x, s]$ such that

$$a(x)b(s) = q_1(x, s)f_1(x, s) + \dots + q_k(x, s)f_k(x, s).$$

Output: $b(s), a(x), q_1(x, s), \dots, q_k(x, s)$; then $b(s)$ is the monic generator of $\mathcal{J} \cap K[s]$.

Theorem 4.6 *The above algorithm is correct.*

Proof: Let us first show the correctness of step (1): This is a parameter version of standard basis computation by homogenization (cf. [La]). Let $\mathrm{lexp}(f(x_0))$ be the leading exponent of $f(x_0) \in K[x_0, x, s]$ with respect to $>$. Then the condition for g stated in the step (1) is equivalent to $\mathrm{lexp}(g^h) = (i, \mu, 0)$ with some $i, \mu \in \mathbf{N}$. Thus g can be chosen from, if any, $g_1(1, x, s), \dots, g_r(1, x, s)$. If such g does not exist, we have $\mathcal{J} \cap K[s] = 0$ since a nonzero element of $\mathcal{J} \cap K[s]$ would satisfy the desired property.

Hence in the sequel, we assume that $g(x, s)$ of step (1) exists. Suppose $\ell_i < \infty$ for any i and put $Q := \overline{J} : b(s)$. Put

$$\mathbf{V}(Q) := \{(x, s) \in \overline{K}^{n+1} \mid f(x, s) = 0 \text{ for any } f \in Q\}.$$

Then we have

$$\begin{aligned} \mathbf{V}(Q) \cap (\{0\} \times \overline{K}) &\subset \mathbf{V}(J) \cap (\{0\} \times \overline{K}) \\ &= \{(0, s) \mid s \in \overline{K},\ f_1(0, s) = \dots = f_k(0, s) = 0\} \\ &= \{(0, s_1), \dots, (0, s_m)\}. \end{aligned} \tag{4.2}$$

Moreover, we have $h_i(x, s)(s - s_i)^{\ell_i} \in \overline{J}$, and hence $h_i(x, s) \in Q$, with some $h_i(x, s) \in \overline{K}[x, s]$ such that $h_i(0, s_i) \neq 0$. This implies, together with (4.2), that $\mathbf{V}(Q) \cap (\{0\} \times \overline{K}) = \emptyset$. In addition, there exists $g(x, s) \in J \subset Q$ whose leading coefficient with respect to s does not vanish at $x = 0$. Hence in view of, e.g., the extension theorem of [CLO, p. 162], there exists some $a(x) \in Q \cap \overline{K}[x]$ with $a(0) \neq 0$. This implies $a(x)b(s) \in \overline{J}$. In particular $b(s)$ belongs to $\overline{\mathcal{J}}' \cap \overline{K}[s]$ with $\overline{\mathcal{J}}' := \overline{K}[x]_0[s]J$.

Let us see that $b(s)$ generates $\overline{\mathcal{J}}' \cap \overline{K}[s]$. If $f(s)$ belongs to $\overline{\mathcal{J}}' \cap \overline{K}[s]$, then $h(x)f(s) \in \overline{J}$ with some $h(x) \in \overline{K}[x]$ with $h(0) \neq 0$. Hence $(s - s_i)^{\ell_i}$ divides $f(s)$ by the definition of ℓ_i. It follows $b(s)$ divides $f(s)$. It also follows that $\mathcal{J}' \cap K[s] = \{0\}$ if $\ell_i = \infty$ for some i.

We have to show $b(x) \in K[s]$. Put $L := K(s_1, \dots, s_m) \subset \overline{K}$ and let $1, \omega_1, \dots, \omega_\nu$ be a basis of L over K and define $\pi(c) := c_0$ for $c = c_0 + c_1\omega_1 + \dots + c_\nu\omega_\nu \in L$ with $c_0, c_1, \dots, c_\nu \in K$. We extend π to the mapping of $L[[x]][s]$ to $K[[x]][s]$. Note that $b(s) \in$

$L[s] \cap L[x]_0[s]J$ since the Gröbner basis computation does not require field extension. Hence we have

$$b(s) = q_1(x,s)f_1(x,s) + \ldots + q_k(x,s)f_k(x,s)$$

with some $q_i(x,s) \in L[[x]][s]$. It follows

$$\pi(b(s)) = \pi(q_1(x,s))f_1(x,s) + \ldots + \pi(q_k(x,s))f_k(x,s) \in \mathcal{J},$$

and hence $\pi(b(s)) \in \mathcal{J}'$ by Lemma 4.4. Thus $b(s)$ divides $\pi(b(s))$ in $L[s]$. Hence both are monic and of the same degree, we have $b(s) = \pi(b(s)) \in K[s]$. Thus we can take $a(x) \in J : b(s) \subset K[x]$ such that $a(0) \neq 0$.

Finally, the inequality $\ell_i \geq \mu_i$ follows from $b(s) \in J(0) = (f_0(s))$. This completes the proof. □

Now let us turn to the computation of P associated with $b_u(s)$. By detecting the FW-Gröbner basis computation and the Gröbner basis computation corresponding to Theorem 4.1, we obtain $P_i \in I$ such that $\psi(P_i) = f_i(x,s)$ in the above notation. Let the F-order of P_i be m_i. By using outputs of Algorithm 4.5, put

$$P := b(t\partial_t) - \frac{1}{a(x)} \sum_{i=1}^{m} q_i(x, t\partial_t) S_i P_i,$$

where $S_i := t^{m_i}$ if $m_i \geq 0$ and $S_i = \partial_t{}^{-m_i}$ if $m_i < 0$. Then we have $P \in F_{-1}(\hat{\mathcal{D}}_{n+1})$ and $(b(t\partial_t) - P)(1 \otimes u) = 0$. How to find a 'good' P, e.g., of minimum order with respect to ∂, remains unsolved.

5 Computation of the Bernstein-Sato polynomial

We retain the notation in the preceding sections. Let $f(x) \in K[x]$ be a polynomial with $f(0) = 0$. (One can suppose $f(x) \in K[[x]]$ with $f(0) = 0$ in the argument below except for the algorithm.) The following argument is due to Malgrange ([M1]). Put $\mathcal{L} = K[[x]][f^{-1}, s]f^s$, where we regard f^s as a free generator. Then $\mathcal{L}$ has a structure of left $\hat{\mathcal{D}}_n[s]$-module defined by

$$\partial_i(g(s)f^{-m}f^s) = \left(\frac{\partial g}{\partial x_i}(s)f^{-m} + (s-m)g(s)\frac{\partial f}{\partial x_i}f^{-m-1}\right)f^s \qquad (i = 1, \ldots, n)$$

for $g(s) \in K[[x]][s]$ and $m \in \mathbf{N}$. Moreover, $\mathcal{L}$ has also a structure of left $\hat{\mathcal{D}}_{n+1}$-module defined by

$$t(g(s)f^s) = g(s+1)f^{s+1}, \qquad \partial_t(g(s)f^s) = -sg(s-1)f^{s-1}$$

for $g(s) \in K[[x]][f^{-1}, s]$. We can make an element $a(t) \in K[[t]]$ operate on $g(s)f^s$ since $f(0) = 0$. It is easy to see that

$$-\partial_t t(g(s)f^s) = sg(s)f^s \quad \text{for any } g(s) \in K[[x]][f^{-1}, s], \tag{5.1}$$

$$(t - f(x))f^s = 0, \tag{5.2}$$

$$\left(\partial_i + \frac{\partial f}{\partial x_i}(x)\partial_t\right)f^s = 0 \quad (i = 1, \ldots, n). \tag{5.3}$$

Put $\mathcal{N} := \hat{\mathcal{D}}_n[s]f^s$ and $\mathcal{M} := \hat{\mathcal{D}}_{n+1}f^s$. Then we have inclusions $\mathcal{N} \subset \mathcal{M} \subset \mathcal{L}$ in view of (5.1). The following lemma is same as Lemme 4.1 of [M1]:

Lemma 5.1 *Put*

$$I := A_{n+1}(t - f(x)) + \sum_{i=1}^{n} A_{n+1} \left(\partial_i + \frac{\partial f}{\partial x_i} \partial_t \right).$$

Then the left ideal $\mathcal{I} = \hat{\mathcal{D}}_{n+1} I$ of $\hat{\mathcal{D}}_{n+1}$ is maximal.

Proposition 5.2 *$\mathcal{M}$ is isomorphic to $\hat{\mathcal{D}}_{n+1}/\mathcal{I}$.*

Proof: Put $\mathcal{I}' := \{P \in \hat{\mathcal{D}}_{n+1} \mid Pf^s = 0\}$. Then we have $\mathcal{M} \simeq \hat{\mathcal{D}}_{n+1}/\mathcal{I}'$. Since $\mathcal{M} \ni f^s \neq 0$, $\mathcal{I}'$ is a proper ideal. The equations (5.2), (5.3) imply $\mathcal{I} \subset \mathcal{I}'$. Since $\mathcal{I}$ is maximal, we must have $\mathcal{I}' = \mathcal{I}$. This completes the proof. □

Corollary 5.3 *For $P(s) \in \hat{\mathcal{D}}_n[s]$, we have $P(s)f^s = 0$ in $\mathcal{N}$ if and only if $P(-\partial_t t) \in \mathcal{I}$.*

Proof: In view of (5.1), we have $P(s)f^s = P(-\partial_t t)f^s$. Hence the assertion follows from Proposition 5.2. □

The (local) b-function (Bernstein-Sato polynomial) $b_f(s)$ of $f(x)$ is the monic polynomial of the least degree $b(s) \in K[s]$ satisfying

$$P(s, x, \partial) f^{s+1} = b(s) f^s \quad \text{in } \mathcal{N} \tag{5.4}$$

with some $P(s) \in \hat{\mathcal{D}}_n[s]$. The monic polynomial of the least degree $b(s)$ satisfying (5.4) with some $P(s) \in A_n[s]$ is denoted by $\tilde{b}_f(s)$. Such $\tilde{b}_f(s)$, and hence $b_f(s)$ also exist (cf. [Be1], [Be2], [Bj], [K1]). By definition $b_f(s)$ divides $\tilde{b}_f(s)$.

In view of Corollary 5.3, the equation (5.4) is equivalent to

$$b(-\partial_t t) - P(-\partial_t t, x, \partial) f \in \mathcal{I}.$$

Since $t - f \in \mathcal{I}$, this is also equivalent to

$$b(-\partial_t t) - P(-\partial_t t, x, \partial) t \in \mathcal{I},$$

and we have $P(-\partial_t t, x, \partial)t \in F_{-1}(\hat{\mathcal{D}}_{n+1})$. On the other hand, suppose $b(s) \in K[s]$ and $Q \in F_{-1}(\hat{\mathcal{D}}_{n+1})$ satisfy

$$(b(t\partial_t) - Q) f^s = 0 \qquad \text{in } \mathcal{M}.$$

Expanding Q in the form

$$Q = \sum_{m=1}^{\infty} Q_m(x, t\partial_t, \partial) t^m,$$

put

$$\rho(Q) := \sum_{m=1}^{\infty} Q_m(x, -s-1, \partial) f^{m-1}.$$

Note that $\rho(Q)$ is well-defined as an element of $\hat{\mathcal{D}}_n[s]$ since $f(0) = 0$. Then we get, in view of Corollary 5.3,

$$(b(-s-1) - \rho(Q)f) f^s = 0 \quad \text{in } \mathcal{N}.$$

In conclusion, the computation of $b_f(s)$ and an associated $P(s) \in \hat{\mathcal{D}}_n[s]$ of (5.4) can be done as follows:

Algorithm 5.4 Input: $f(x) \in K[x]$;

(1) Letting I be the left ideal of A_{n+1} generated by $t - f$ and $\partial_i + (\partial f/\partial x_i)\partial_t$ $(i = 1, \ldots, n)$, compute an FW-Gröbner basis $\mathbf{G}$ of I via F-homogenization;

(2) Compute a Gröbner basis $\mathbf{H}$ of the left ideal generated by $\psi(\mathbf{G}) := \{\psi(P) \mid P \in \mathbf{G}\}$ with respect to an order satisfying (A-1), (A-2), (A-5); In the course of this computation, find $P_1, \ldots, P_m \in \mathbf{G}$ such that $\mathbf{H} \cap K[x, s] = \{\psi(P_1), \ldots, \psi(P_m)\}$;

(3) Compute the outputs $b(s) \in K[s], a(x) \in K[x], q_1(x, s), \ldots, q_s(x, s) \in K[x, s]$ of Algorithm 4.5 with $\mathbf{H} \cap K[x, s]$ as inputs;

(4) Put $Q := a(x)b(t\partial_t) - \sum_{i=1}^{m} q_i(x, t\partial_t)S_i P_i \in F_{-1}(A_{n+1})$, where $S_i := t^{m_i}$ if $m_i := \mathrm{ord}_F(P_i) \geq 0$ and $S_i := {\partial_t}^{-m_i}$ otherwise;

Output: $b_f(s) := b(-s-1)$ and $P(s) := (1/a(x))\rho(Q)$.

Remark 5.5 (1) The step (1) of Algorithm 4.5, which is called in the above algorithm, can be skipped since the existence of $\tilde{b}_f(s)$ is assured by Bernstein ([Be2]).

(2) If K is a subfield of $\mathbf{C}$, the fact that the roots of $b_f(s)$ are rational (Kashiwara [K1]) makes the steps (3) and (4) of Algorithm 4.5 considerably easier since there is no need of field extension.

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