Minimal free resolutions of analytic D-modules

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We introduce the notion of minimal free resolution for a filtered module over the ring $\mathcal{D}$ of analytic differential operators. A module over $\mathcal{D}$ corresponds to a system of linear partial differential equations with analytic coefficients. Hence a filtered free resolution of a filtered $\mathcal{D}$-module is essential in studying homological properties of linear partial differential equations. We also give some examples of minimal free resolution of the $\mathcal{D}$-module generated by the reciprocal $1/f$ of a polynomial f with a singularity at the origin. This defines analytic invariants attached to hypersurface singularities.

1 Filtered modules over the ring of analytic differential operators

Let $\mathcal{D} = \mathcal{D}_n$ be the ring of differential operators with convergent power series coefficients in n variables $x = (x_1, \dots, x_n)$. An element P of $\mathcal{D}$ is written as a finite sum

$$P = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(x) \partial^\alpha$$

with $a_\alpha(x)$ belonging to the ring of convergent power series $\mathcal{O} = \mathbb{C}\{x_1, \dots, x_n\}$, where we use the notation $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ with $\partial_i = \partial/\partial x_i$. Then the order of P is defined by

$$\operatorname{ord} P := \max\{|\alpha| = \alpha_1 + \cdots + \alpha_n \mid a_\alpha(x) \neq 0\}.$$

A presentation of a finitely generated left $\mathcal{D}$-module M is an exact sequence

$$\mathcal{D}^{r_1} \xrightarrow{\varphi_1} \mathcal{D}^{r_0} \xrightarrow{\varphi_0} M \to 0$$

of left D-modules. The homomorphism φ_1 is defined by

$$\varphi_1 : \mathcal{D}^{r_1} \ni U = (U_1, \dots, U_{r_1}) \longmapsto UP \in D^{r_0}$$

with an $r_1 \times r_0$ matrix $P = (P_{ij})$ whose elements are in $\mathcal{D}$. Hence we often identify the homomorphism φ_1 with the matrix P. It is the starting point of the D-module theory to regard M as a system of linear differential equations

$$\sum_{j=1}^{r_0} P_{ij} u_j = 0 \qquad (i = 1, \dots, r_1)$$

for unknown functions $u_1, \dots, u_{r_0}$.

We define the order filtration on $\mathcal{D}$ by

$$F_k(\mathcal{D}) := \{P \in \mathcal{D} \mid \operatorname{ord} P \le k\} \qquad (k \in \mathbb{Z}).$$

A filtered free module is the free module $\mathcal{D}^r$ equipped with the filtration

$$F_k[\mathbf{m}](\mathcal{D}^r) := F_{k-m_1}(\mathcal{D}) \oplus \cdots \oplus F_{k-m_r}(\mathcal{D})$$

with some $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$, which we call the shift vector. Let M be a finitely generated left $\mathcal{D}$-module. Then a filtration on M is a family $\{F_k(M)\}_{k\in\mathbb{Z}}$ of $\mathbb{C}$-subspaces of M such that

$$F_k(M) \subset F_{k+1}(M), \quad \bigcup_{k\in\mathbb{Z}} F_k(M) = M, \quad F_m(\mathcal{D})F_k(M) \subset F_{k+m}(M).$$

A filtration $\{F_k(M)\}_{k\in\mathbb{Z}}$ on M is called a good filtration if there exist $r \in \mathbb{N}$, $\mathbf{m} \in \mathbb{Z}^r$, and a homomorphism $\varphi : \mathcal{D}^r \to M$ of left $\mathcal{D}$-modules such that

$$\varphi(F_k[\mathbf{m}](\mathcal{D}^r)) = F_k(M) \qquad (\forall k \in \mathbb{Z}).$$

If $\{F_k(M)\}_{k\in\mathbb{Z}}$ is a good filtration, the induced filtration $\{F_k(M) \cap N\}_{k\in\mathbb{Z}}$ is good for any submodule N of M.

A filtered free resolution of a filtered $\mathcal{D}$-module M is an exact sequence

$$\cdots \xrightarrow{\varphi_3} \mathcal{D}^{r_2} \xrightarrow{\varphi_2} \mathcal{D}^{r_1} \xrightarrow{\varphi_1} \mathcal{D}^{r_0} \xrightarrow{\varphi_0} M \to 0$$

of left $\mathcal{D}$-modules with shift vectors $\mathbf{m}_i \in \mathbb{Z}^{r_i}$ $(i \ge 0)$ such that

$$\cdots \xrightarrow{\varphi_3} F_k[\mathbf{m}_2](\mathcal{D}^{r_2}) \xrightarrow{\varphi_2} F_k[\mathbf{m}_1](\mathcal{D}^{r_1}) \xrightarrow{\varphi_1} F_k[\mathbf{m}_0](\mathcal{D}^{r_0}) \xrightarrow{\varphi_0} F_k(M) \to 0$$

is exact for any $k \in \mathbb{Z}$. The graded ring of $\mathcal{D}$ is defined by

$$\operatorname{gr}(\mathcal{D}) = \bigoplus_{k\ge 0} F_k(\mathcal{D})/F_{k-1}(\mathcal{D}) \simeq \mathcal{O}[\xi] = \mathcal{O}[\xi_1, \dots, \xi_n],$$

where $\mathcal{O}[\xi]$ denotes the polynomial ring in the commutative variables $\xi = (\xi_1, \dots, \xi_n)$ with coefficients in $\mathcal{O}$.

If the order of the differential operator $P = \sum_{\alpha\in\mathbb{N}^n} a_\alpha(x)\partial^\alpha$ is k, then its principal symbol is defined to be

$$\sigma(P) = \sigma_k(P) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha \in \operatorname{gr}(\mathcal{D}).$$

The graded ring $\mathrm{gr}(\mathcal{D})$ has a natural structure of commutative graded ring

$$\mathrm{gr}(\mathcal{D}) = \bigoplus_{k\geq 0} \mathrm{gr}(\mathcal{D})_k, \qquad \mathrm{gr}(\mathcal{D})_k := F_k(\mathcal{D})/F_{k-1}(\mathcal{D}) \simeq \mathcal{O}[\xi]_k,$$

where $\mathcal{O}[\xi]_k$ is the subspace of $\mathcal{O}[\xi]$ consisting of homogeneous elements of degree k with respect to ξ. If M is a filtered $\mathcal{D}$-module, then

$$\mathrm{gr}(M) := \bigoplus_{k\in\mathbb{Z}} F_k(M)/F_{k-1}(M)$$

is a graded $\mathrm{gr}(\mathcal{D})$-module. In particular, the graded module of the filtered free module $(\mathcal{D}^r, F_\bullet[\mathbf{m}])$ is defined to be

$$\mathrm{gr}[\mathbf{m}](\mathcal{D}^r) = \sum_{k\in\mathbb{Z}} F_k[\mathbf{m}](\mathcal{D}^r)/F_{k-1}[\mathbf{m}](\mathcal{D}^r).$$

2 Minimal filtered free resolutions

The graded ring $\mathrm{gr}(\mathcal{D})$ has a unique maximal graded ideal

$$\mathrm{gr}(\mathcal{D})x_1 + \cdots + \mathrm{gr}(\mathcal{D})x_n + \mathrm{gr}(\mathcal{D})\xi_1 + \cdots + \mathrm{gr}(\mathcal{D})\xi_n = (\mathcal{O}x_1 + \cdots + \mathcal{O}x_n) + \bigoplus_{k\geq 1} \mathrm{gr}(\mathcal{D})_k.$$

Hence the notion of minimal free resolution of a graded $\mathrm{gr}(\mathcal{D})$-module makes sense. A minimal free resolution of a graded $\mathrm{gr}(\mathcal{D})$-module M' is an exact sequence

$$\cdots \xrightarrow{\varphi_3} \mathrm{gr}[\mathbf{m}_2](\mathcal{D}^{r_2}) \xrightarrow{\varphi_2} \mathrm{gr}[\mathbf{m}_1](\mathcal{D}^{r_1}) \xrightarrow{\varphi_1} \mathrm{gr}[\mathbf{m}_0](\mathcal{D}^{r_0}) \xrightarrow{\varphi_0} M' \to 0$$

of graded $\mathrm{gr}(\mathcal{D})$-modules with $\mathbf{m}_i \in \mathbb{Z}^{r_i}$ $(i \geq 0)$ such that each φ_i is homogeneous of degree 0 (with respect to ξ) and does not contain invertible elements in $\mathcal{O}$ as an entry for $i \geq 1$, or equivalently that $\{\varphi_i(1,0,\ldots,0),\ \ldots,\ \varphi_i(0,\ldots,0,1)\}$ is a minimal set of generators of $\mathrm{Ker}\,\varphi_{i-1}$ for all $i \geq 1$ and $\{\varphi_0(1,0,\ldots,0),\ \ldots,\ \varphi_0(0,\ldots,0,1)\}$ is a minimal set of generators of M'.

It is well-known that a minimal free resolution is unique up to isomorphism (see e.g., [E]). In particular, the ranks r_i and the shift vectors $\mathbf{m}_i$ (up to permutation of their entries) are invariants of M'.

Definition 1 (minimal filtered free resolution) Let M be a finitely generated left $\mathcal{D}$-module with a filtration $\{F_k(M)\}_{k\in\mathbb{Z}}$. A filtered free resolution

$$\cdots \xrightarrow{\psi_3} \mathcal{D}^{r_2} \xrightarrow{\psi_2} \mathcal{D}^{r_1} \xrightarrow{\psi_1} \mathcal{D}^{r_0} \xrightarrow{\psi_0} M \to 0$$

of M (with shift vectors $\mathbf{m}_i \in \mathbb{Z}^{r_i}$) is called a *minimal filtered free resolution* of M if the induced exact sequence

$$\cdots \xrightarrow{\overline{\psi}_3} \mathrm{gr}[\mathbf{m}_1](\mathcal{D}^{r_2}) \xrightarrow{\overline{\psi}_2} \mathrm{gr}[\mathbf{m}_1](\mathcal{D}^{r_1}) \xrightarrow{\overline{\psi}_1} \mathrm{gr}[\mathbf{m}_0](\mathcal{D}^{r_0}) \xrightarrow{\overline{\psi}_0} \mathrm{gr}(M) \to 0$$

is a minimal free resolution of $\mathrm{gr}(M)$.

By using standard arguments in commutative algebra, we can easily prove

Theorem 1 *Let M be a finitely generated left $\mathcal{D}$-module with a good filtration. Then a minimal filtered free resolution of M exists and is unique up to isomorphism; i.e., if there are two minimal filtered free resolutions*

$$\cdots \longrightarrow \mathcal{D}^{r_3} \xrightarrow{\psi_3} \mathcal{D}^{r_2} \xrightarrow{\psi_2} \mathcal{D}^{r_1} \xrightarrow{\psi_1} \mathcal{D}^{r_0} \xrightarrow{\psi_0} M \to 0$$

with shift vectors $\mathbf{m}_i$ $(i \geq 0)$ and

$$\cdots \longrightarrow \mathcal{D}^{r'_3} \xrightarrow{\psi'_3} \mathcal{D}^{r'_2} \xrightarrow{\psi'_2} \mathcal{D}^{r'_1} \xrightarrow{\psi'_1} \mathcal{D}^{r'_0} \xrightarrow{\psi'_0} M \to 0$$

with shift vectors $\mathbf{m}'_i$ $(i \geq 0)$, then there exist $\mathcal{D}$-isomorphisms $\theta_i : \mathcal{D}^{r_i} \to \mathcal{D}^{r'_i}$ satisfying $\theta_i(F_k[\mathbf{m}_i](\mathcal{D}^{r_i})) = F_k[\mathbf{m}'_i](\mathcal{D}^{r'_i})$ for any $k \in \mathbb{Z}$, such that the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow \mathcal{D}^{r_3} & \xrightarrow{\psi_3} & \mathcal{D}^{r_2} & \xrightarrow{\psi_2} & \mathcal{D}^{r_1} & \xrightarrow{\psi_1} & \mathcal{D}^{r_0} & \xrightarrow{\psi_0} & M \\
\theta_3 \downarrow & & \theta_2 \downarrow & & \theta_1 \downarrow & & \theta_0 \downarrow & & \| \\
\cdots \longrightarrow \mathcal{D}^{r'_3} & \xrightarrow{\psi'_3} & \mathcal{D}^{r'_2} & \xrightarrow{\psi'_2} & \mathcal{D}^{r'_1} & \xrightarrow{\psi'_1} & \mathcal{D}^{r'_0} & \xrightarrow{\psi'_0} & M
\end{array}$$

is commutative. In particular, we have $r_i = r'_i$, and $\mathbf{m}_i$ and $\mathbf{m}'_i$ coincide up to permutation of their entries.

Let I be a left ideal of $\mathcal{D}$. A subset $G = \{P_1, \dots, P_s\}$ of I is called a *minimal set of involutive generators* of I if $\sigma(G) := \{\sigma(P_1), \dots, \sigma(P_s)\}$ is a minimal set of homogeneous generators of the graded $\mathrm{gr}(\mathcal{D})$-module

$$\mathrm{gr}(I) := \bigoplus_{k \in \mathbb{Z}} (F_k(\mathcal{D}) \cap I)/(F_{k-1}(\mathcal{D}) \cap I).$$

The theorem above implies

Corollary 1 *In the notation above, suppose that $G = \{P_1, \dots, P_r\}$ and $G' = \{P'_1, \dots, P'_{r'}\}$ are minimal sets of involutive generators of a left ideal I of $\mathcal{D}$ with*

$$\mathrm{ord}\, P_1 \leq \mathrm{ord}\, P_2 \leq \cdots \leq \mathrm{ord}\, P_r, \qquad \mathrm{ord}\, P'_1 \leq \mathrm{ord}\, P'_2 \leq \cdots \leq \mathrm{ord}\, P'_r.$$

Then we have $r = r'$ and $\mathrm{ord}\, P_i = \mathrm{ord}\, P'_i$ for $i = 1, \dots, r$. Moreover, there exists an invertible $r \times r$ matrix $U = (U_{ij})$ with entries in $\mathcal{D}$ satisfying

$$P_i = \sum_{j=1}^{r} U_{ij} P'_j, \qquad \max\{\mathrm{ord}\, U_{ij} + \mathrm{ord}\, P'_j \mid j = 1, \dots, r\} = \mathrm{ord}\, P_i \qquad (i = 1, \dots, r).$$

3 Minimal filtered free resolutions of $\mathrm{Ann}_{\mathcal{D}} f^{-1}$

Let $f \in \mathbb{C}[x]$ be a polynomial in n variables $x = (x_1, \dots, x_n)$ and consider the annihilating ideal

$$\mathrm{Ann}_{\mathcal{D}} f^{-1} = \{P \in \mathcal{D} \mid Pf^{-1} = 0\}.$$

This is equipped with the filtration $\{\mathrm{Ann}_{\mathcal{D}} f^{-1} \cap F_k(\mathcal{D})\}_{k \geq 0}$.

The simplest case is when f is a so-called Koszul free divisor; i.e., $\mathrm{Ann}_{\mathcal{D}} f^{-1}$ is generated by first order operators whose principal symbols constitute a regular sequence in $\mathcal{O}[\xi]$. Then a minimal filtered free resolution of $\mathrm{Ann}_{\mathcal{D}} f^{-1}$ is of the form

$$0 \to \mathcal{D} \to \mathcal{D}^n \to \mathcal{D}^{n(n-1)/2} \to \cdots \to \mathcal{D}^n \to \mathrm{Ann}_{\mathcal{D}} f^{-1} \to 0$$

with shift vectors

$$\mathbf{m}_1 = (1, \dots, 1), \quad \mathbf{m}_2 = (2, \dots, 2), \quad \cdots, \quad \mathbf{m}_n = (n).$$

If f is non-singular, or has normal crossing singularity, then it is Koszul free.

The following examples are computed by using a program Kan of N. Takayama, which realizes the algorithm of [OT]. This method produces minimal filtered free resolutions of modules over the (homogenized) Weyl algebra with respect to the weight vector $(0, \dots, 0; 1, \dots, 1)$. In each example below, we can verify that the output is also a minimal filtered free resolution over $\mathcal{D}$.

Example 1 Put $f = x^3 - y^2$ with two variables x and y. Then f is a Koszul free divisor. A minimal free resolution of $\mathrm{Ann}_{\mathcal{D}} f^{-1}$ is given by

$$0 \to \mathcal{D} \xrightarrow{\psi_2} \mathcal{D}^2 \xrightarrow{\psi_1} \mathrm{Ann}_{\mathcal{D}} f^{-1} \to 0$$

with shift vectors

$$\mathbf{m}_1 = (1, 1), \quad \mathbf{m}_2 = (2)$$

and homomorphisms defined by matrices

$$\psi_1 = \begin{pmatrix} 2x\partial_x + 3y\partial_y + 6 \\ -3x^2\partial_y - 2y\partial_x \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} 3x^2\partial_y + 2y\partial_x & 2x\partial_x + 3y\partial_y + 5 \end{pmatrix}$$

with $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$.

Example 2 Put $f = x^4 + y^5 + xy^4$ with two variables x, y. A minimal free resolution of $\mathrm{Ann}_{\mathcal{D}} f^{-1}$ is given by

$$0 \to \mathcal{D}^2 \xrightarrow{\psi_2} \mathcal{D}^3 \xrightarrow{\psi_1} \mathrm{Ann}_{\mathcal{D}} f^{-1} \to 0$$

with shift vectors

$$\mathbf{m}_1 = (1, 1, 2), \quad \mathbf{m}_2 = (2, 2)$$

and homomorphisms

$$\psi_1 = \begin{pmatrix} -16x^2\partial_x - 20xy\partial_x - 12xy\partial_y - 16y^2\partial_y - 64x - 80y \\ P_2 \\ P_3 \end{pmatrix}$$

with

$$\begin{aligned}
P_2 = &- 64xy^2\partial_x - 16y^3\partial_x - 48y^3\partial_y + 500xy\partial_x + 16x^2\partial_y - 20xy\partial_y + 400y^2\partial_y \\
&- 256y^2 + 2000y, \\
P_3 = &- 262144y^3\partial_x^2 + 262144y^3\partial_x\partial_y - 2048000xy\partial_x^2 + 573440xy\partial_x\partial_y - 1638400y^2\partial_x\partial_y \\
&- 196608x^2\partial_y^2 + 32768xy\partial_y^2 + 393216y^2\partial_y^2 - 1835008xy\partial_x + 589824y^2\partial_x \\
&- 1376256y^2\partial_y + 15237120x\partial_x - 10240000y\partial_x - 425984x\partial_y + 14630912y\partial_y \\
&- 7340032y + 60948480
\end{aligned}$$

and

$$\psi_2 = \begin{pmatrix} P_{11} & -16384x\partial_x - 16384y\partial_x - 45056 & y \\ P_{21} & 4096x\partial_x + 12288x\partial_y + 16384y\partial_y + 53248 & x \end{pmatrix}$$

with

$$\begin{aligned}
P_{11} &= 65536y^2\partial_x - 512000y\partial_x - 16384x\partial_y + 24576y\partial_y + 20480, \\
P_{21} &= -16384y^2\partial_x - 49152y^2\partial_y + 4096x\partial_y + 409600y\partial_y - 212992y + 1331200.
\end{aligned}$$

From this resolution, we know that $\mathrm{Ann} f^{-1}$ cannot be generated by first order operators.

Example 3 Put $f = xyz$ with three variables x, y, z. Then f has a normal crossing singularity. A minimal filtered free resolution of $\mathrm{Ann}_{\mathcal{D}} f^{-1}$ is given by

$$0 \to \mathcal{D} \xrightarrow{\psi_3} \mathcal{D}^3 \xrightarrow{\psi_2} \mathcal{D}^3 \xrightarrow{\psi_1} \mathrm{Ann}_{\mathcal{D}} f^{-1} \to 0$$

with shift vectors

$$\mathbf{m}_1 = (1,1,1), \quad \mathbf{m}_2 = (2,2,2), \quad \mathbf{m}_3 = (3)$$

and homomorphisms

$$\psi_1 = \begin{pmatrix} -x\partial_x - 1 \\ z\partial_z + 1 \\ y\partial_y + 1 \end{pmatrix}, \qquad \psi_2 = \begin{pmatrix} -z\partial_z - 1 & -x\partial_x - 1 & 0 \\ -y\partial_y - 1 & 0 & -x\partial_x - 1 \\ 0 & y\partial_y + 1 & -z\partial_z - 1 \end{pmatrix},$$

$$\psi_3 = \begin{pmatrix} -y\partial_y - 1 & z\partial_z + 1 & -x\partial_x - 1 \end{pmatrix}.$$

Example 4 Put $f = x^2 + y^2 + z^2$ with variables x, y, z. A minimal filtered free resolution of $\mathrm{Ann}_{\mathcal{D}} f^{-1}$ is given by

$$0 \to \mathcal{D}^2 \xrightarrow{\psi_3} \mathcal{D}^5 \xrightarrow{\psi_2} \mathcal{D}^4 \xrightarrow{\psi_1} \mathrm{Ann}_{\mathcal{D}} f^{-1} \to 0$$

with shift vectors

$$\mathbf{m}_1 = (1,1,1,1), \quad \mathbf{m}_2 = (1,2,2,2,2), \quad \mathbf{m}_3 = (3,2)$$

and homomorphisms

$$\psi_1 = \begin{pmatrix} z\partial_y - y\partial_z \\ z\partial_x - x\partial_z \\ y\partial_x - x\partial_y \\ -x\partial_x - y\partial_y - z\partial_z - 2 \end{pmatrix},$$

$$\psi_2 = \begin{pmatrix} -x & y & -z & 0 \\ -y\partial_y - z\partial_z - 2 & -x\partial_y & x\partial_z & -z\partial_y + y\partial_z \\ \partial_x & -\partial_y & \partial_z & 0 \\ -y\partial_x & -x\partial_x - z\partial_z - 2 & -y\partial_z & -z\partial_x + x\partial_z \\ x\partial_z & -y\partial_z & -x\partial_x - y\partial_y - 1 & -y\partial_x + x\partial_y \end{pmatrix},$$

$$\psi_3 = \begin{pmatrix} -\partial_z^2 & -\partial_x & -z\partial_z - 1 & \partial_y & -\partial_z \\ x\partial_x + y\partial_y + 1 & -x & x^2 + y^2 & y & -z \end{pmatrix}.$$

Example 5 Put $f = x^3 - y^2z^2$ with variables x, y, z. Then f has non-isolated singularities. A minimal filtered free resolution of $\mathrm{Ann}_{\mathcal{D}} f^{-1}$ is given by

$$0 \to \mathcal{D}^2 \xrightarrow{\psi_3} \mathcal{D}^5 \xrightarrow{\psi_2} \mathcal{D}^4 \xrightarrow{\psi_1} \mathrm{Ann}_{\mathcal{D}} f^{-1} \to 0$$

with shift vectors

$$\mathbf{m}_1 = (1,1,1,1), \quad \mathbf{m}_2 = (1,2,2,2,2), \quad \mathbf{m}_3 = (2,3)$$

and homomorphisms

$$\psi_1 = \begin{pmatrix} -y\partial_y + z\partial_z \\ -2x\partial_x - 3z\partial_z - 6 \\ -2yz^2\partial_x - 3x^2\partial_y \\ -2y^2z\partial_x - 3x^2\partial_z \end{pmatrix},$$

$$\psi_2 = \begin{pmatrix} -3x^2 & 0 & y & -z \\ 6yz^2\partial_x + 9x^2\partial_y & 2yz^2\partial_x + 3x^2\partial_y & -2x\partial_x - 3y\partial_y - 5 & 0 \\ 0 & -2y^2z\partial_x - 3x^2\partial_z & 0 & 2x\partial_x + 3z\partial_z + 5 \\ 2x\partial_x + 3z\partial_z + 6 & -y\partial_y + z\partial_z & 0 & 0 \\ 2yz\partial_x & 0 & \partial_z & -\partial_y \end{pmatrix},$$

$$\psi_3 = \begin{pmatrix} 2x\partial_x + 3y\partial_y + 3z\partial_z + 2 & y & z & 3x^2 & -3yz \\ 3\partial_y\partial_z & \partial_z & \partial_y & -2yz\partial_x & 2x\partial_x + 2 \end{pmatrix}.$$

Example 6 Put $f = xy(x+y)(xz+y)$ with variables x, y, z. This polynomial was studied in [CU]. A minimal filtered free resolution of $\mathrm{Ann}_{\mathcal{D}} f^{-1}$ is given by

$$0 \to \mathcal{D}^2 \xrightarrow{\psi_3} \mathcal{D}^5 \xrightarrow{\psi_2} \mathcal{D}^4 \xrightarrow{\psi_1} \mathrm{Ann}_{\mathcal{D}} f^{-1} \to 0$$

with shift vectors

$$\mathbf{m}_1 = (1,1,1,2), \quad \mathbf{m}_2 = (2,2,3,2,2), \quad \mathbf{m}_3 = (3,3)$$

and homomorphisms

$$\psi_1 = \begin{pmatrix} -9x\partial_x - 9y\partial_y - 36 \\ 36xz\partial_z + 36y\partial_z + 36x \\ -36xy\partial_y - 36y^2\partial_y - 36yz\partial_z + 36y\partial_z - 36x - 108y \\ 432z^2\partial_z^2 + 432y\partial_x\partial_z - 432y\partial_y\partial_z - 432z\partial_z^2 + 864z\partial_z - 1296\partial_z \end{pmatrix},$$

$$\psi_2 = \begin{pmatrix} -48y\partial_z & 12z\partial_z - 12\partial_z & 12\partial_z & -x \\ P_{21} & -12x\partial_x - 12y\partial_x - 24 & 12z\partial_z + 12 & y \\ P_{31} & -12z\partial_x\partial_z + 12\partial_x\partial_z & -12\partial_x\partial_z & -y\partial_y - 3 \\ -4xz\partial_z - 4y\partial_z - 4x & -x\partial_x - y\partial_y - 3 & 0 & 0 \\ P_{51} & 0 & x\partial_x + y\partial_y + 3 & 0 \end{pmatrix}$$

with

$$\begin{aligned} P_{21} &= -48xz\partial_z - 48yz\partial_z - 48y\partial_z - 48x - 48y, \\ P_{31} &= -48z^2\partial_z^2 + 48y\partial_y\partial_z + 48z\partial_z^2 - 96z\partial_z + 144\partial_z, \\ P_{51} &= -4xy\partial_y - 4y^2\partial_y - 4yz\partial_z + 4y\partial_z - 4x - 12y \end{aligned}$$

and

$$\psi_3 = \begin{pmatrix} y\partial_y + 3 & 0 & -x & 12z\partial_z - 12\partial_z & -12\partial_z \\ -y\partial_x & -x\partial_x - y\partial_y - 3 & -y & 12x\partial_x + 12y\partial_x + 24 & 12z\partial_z + 12 \end{pmatrix}.$$

As was shown in [CU], $\mathrm{Ann} f^{-1}$ can be generated by the first three components of ψ_1, which are first order operators but are not involutive.

References

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