A C_n-move for a knot and the coefficients of the Conway polynomial

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ABSTRACT

It is shown that two knots can be transformed into each other by C_n-moves if and only if they have the same Vassiliev invariants of order less than n. Consequently, a C_n-move cannot change the Vassiliev invariants of order less than n and can change those of order more than or equal to n. In this paper, we consider the coefficient of the Conway polynomial as a Vassiliev invariant and show that a C_n-move can change the nth coefficient of the Conway polynomial by ± 2, or 0. And for the $2m$th coefficient($2m > n$), it can change by p or $p+1$ for any given integer p.

Keywords: Vassiliev invariant, Conway polynomial, C_n-move

1. Introduction

When we have a knot invariant v which takes values in some abelian group, we can define an invariant of singular knots by the Vassiliev skein relation:

$$v(K_D) = v(K_+) - v(K_-),$$

where *a singular knot* is an immersion of a circle into R^3 whose singularities are double points only and K_D, K_+ and K_- denote the diagrams of singular knots which are identical except near one point as is shown in Fig. 1.

An invariant v is called *a Vassiliev invariant of order* n and denoted by v_n, if n is the smallest integer such that v vanishes on all singular knots with more than n double points([3]).

A standard C_n-move is a local move depicted in Fig. 2 and *a C_1-move* is defined as a crossing change. M. N. Goussarov([4]) and K. Habiro([6]) showed independently

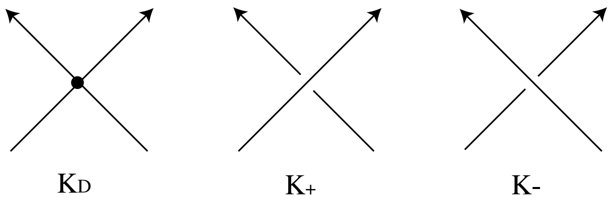


Fig. 1

the following theorem.

Theorem 1.1([4], [6]). *Two knots can be transformed into each other by a finite sequence of standard C_n-moves if and only if they have the same Vassiliev invariants of order less than n.*

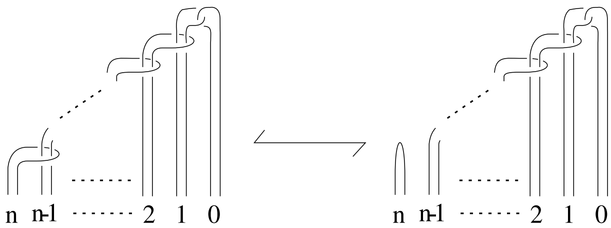


Fig. 2

C_n-moves are originally defined by Habiro in [5]. In [15] and [19], they are defined as a family of local moves. It is known that any kind of C_n-moves can be realized by a finite sequence of standard C_n-moves.

From Theorem 1.1, a C_n-move cannot change the Vassiliev invariants of order less than n and it can change Vassiliev invariants of order more than or equal to n. We consider the coefficient of the Conway polynomial as a Vassiliev invariant. Let $a_n(K)$ be the coefficient of z^n in the Conway polynomial for a knot K.

A C_2-move is the same move as a delta move([8], [10]). By the result of M. Okada in [17], if the knot K' is obtained from the knot K by a single delta move, then $a_2(K') - a_2(K) = \pm 1$. Therefore a C_2-move can change the value of a_2 by ± 1.

In the case that $n \geq 3$, it is known the following theorem.

Theorem 1.2([9]). *If a link L' is obtained from a link L by a standard C_n-move $(n \geq 3)$, then*

$$a_n(L') - a_n(L) \equiv 0 \quad (mod \quad 2).$$

Theorem 1.2 shows that a C_n-move changes the value of a_n by even number. We consider the case of knots and show Theorem 1.3. Since the Conway polynomial for a knot has only terms of even degree, then we consider a C_{2n}-move for a knot.

Theorem 1.3. *If a knot K' is obtained from a knot K by a standard C_{2n}-move $(n \geq 2)$, then*

$$a_{2n}(K') - a_{2n}(K) = 0 \quad \textit{or} \quad \pm 2.$$

If a knot K can be transformed into K' by standard C_n-moves, we denote the minimum number of C_n-moves needed to transform K into K' by $d_{C_n}(K, K')$ and call it *the C_n-distance between K and K'*.

Corollary 1.4. *If a knot K' is obtained from a knot K by standard C_{2n}-moves $(n \geq 2)$, then*

$$d_{C_{2n}}(K, K') \geq \frac{|a_{2n}(K') - a_{2n}(K)|}{2}.$$

When we perform a C_n-move, for a coefficient of higher degree $a_{2m}(K)$ $(2m > n)$, the second author showed the following theorem in [21].

Theorem 1.5([21]). *Let n be a natural number. For any integer sequence $(p_n, p_{n+1}, \cdots, p_\ell)$, there exists a pair of knots K and K' with $d_{C_n}(K, K') = 1$ satisfying the following:*

$$\begin{aligned} a_{2m}(K') - a_{2m}(K) &= p_m \ (n \leq m \leq \ell) \ \textit{and} \\ a_{2m}(K') - a_{2m}(K) &= 0 \quad (\ell < m). \end{aligned}$$

From Theorem 1.5, a C_n-move can change a_{2m} $(n \leq m)$ by any given integer. In this paper, we also consider a_{2m} in the case $n < 2m < 2n$ and obtain Theorem 1.6.

Theorem 1.6. *Let m and n be a pair of natural numbers with $2m > n$. For any integer p, there exists a pair of knots K and K' with $d_{C_n}(K, K') = 1$ satisfying*

$$a_{2m}(K') - a_{2m}(K) = p \ \textit{or} \ p + 1.$$

2. C_n-moves, Jacobi diagrams and the Conway polnomial

A tangle T is a disjoint union of properly embedded arcs in the unit 3-ball B^3. A tangle T is *trivial* if there exisits a properly embedded disk in B^3 containing T. *A local move* is a pair of trivial tangles (T_1, T_2) with $\partial T_1 = \partial T_2$ such that for each component t of T_1 there exists a component u of T_2 with $\partial t = \partial u$. Let (T_1, T_2) be a local move, t_1 a component of T_1 and t_2 a component of T_2 such that $\partial t_1 = \partial t_2$. Replacing t_1 and t_2 by hooked arcs in Fig. 3, we obtain a new kind of local move. This local move is called a *double of* (T_1, T_2) *with respect to the components* t_1 *and* t_2.

A C_1*-move* is a local move as illustrated in Fig. 4. *A* C_{k+1}*-move* is a double of a C_k-move. Then, there exist some kinds of C_n-move and any kind of C_n-move is realized by a finite sequence of standard C_n-moves. For details, refer to [15] or [19].

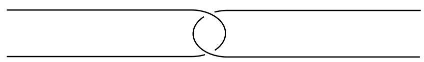


Fig. 3

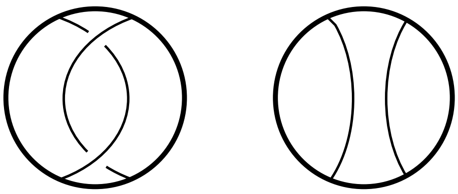


Fig. 4

By K^m, we denote a singular knot with m double points. From the definition of the Vassiliev invariant, $v_m(K^m)$ does not change by a crossing change and it is determined by the positions of double points on K^m. To show the positions of double points, the notion of a chord diagram is introduced in [3]. *A chord diagram of order* n is an oriented circle with n chords. By connecting the preimages of each double point by a chord, we may associate the chord diagram to a singular knot. The value of v_n for a chord diagram of order n is defined as the value for a singular knot with n double points that is associated with the chord diagram.

Chord diagrams are generalized to Jacobi diagrams in [1]. *A Jacobi diagram of order* n is a trivalent graph with $2n$ vertices. It is a union of a circle and an internal graph G. The circle is oriented and the other edges are all unoriented. Each trivalent vertex on G has an orientation, that is a cyclic ordering of the edges incident to it. In the additive group generated by the Jacobi diagrams of order n, the relation in Fig. 5 is called *the STU relation. The IHX relation* in Fig. 6 and *the* *antisymmetry relation* in Fig. 7 can be obtained as a consequence of STU relations.

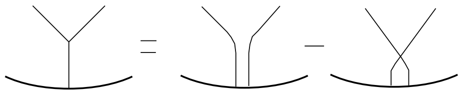


Fig. 5

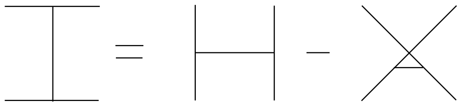


Fig. 6

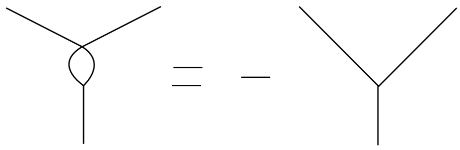


Fig. 7

A one branch tree diagram T is a special kind of Jacobi diagram whose internal graph G is isomorphic to a standard n-tree in Fig. 8 preserving the vertex orientations([13]). Label the branches of the standard n-tree as in Fig. 8. We may labell the branches of G under the isomorophism between G and the standard n-tree. And number the vertices of the circle of T by 0, 1, 2, ..., n in the counterclockwise direction such that the end the branch 0 of G is numbered by 0. The correspondence between the labels of branches of G and the numbers of their end points on the circle determines a permutation $\sigma \in S_n$. Conversely, if a permutation $\sigma \in S_n$ is given, a unique one branch tree diagram T can be constructed. Then we denote a one branch tree diagram by T_σ. By STU-relations, a one branch tree diagram can be expressed as a signed sum of chord diagrams. The value of v_n for a one branch tree diagram of order n is defined as the signed sum of the values for chord diagrams.

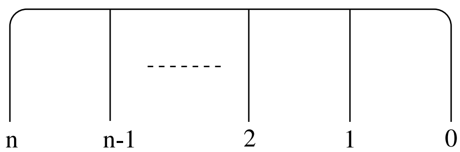


Fig. 8

A one branch tree diagram is closely related to a standard C_n-move.

Theorem 2.1([16]). *Let v_n be a Vassiliev invariant of order n. If a knot K' is obtained from a knot K by a single standard C_n-move, then*

$$v_n(K') - v_n(K) = \pm v_n(T_\sigma),$$

where T_σ is a one branch tree diagram of order n.

In Theorem 2.1, the one branch tree diagram T_σ is determined by the positions of arcs in the C_n-move on a knot K and the sign of the formula depends only on the orientations of arcs in the C_n-move.

We can define a Vassiliev invariant for μ component links as for knots. A singular μ component link is an immersion of μ circles into R^3. And we can also define a chord diagram and a Jacobi diagram for a link. A chord diagram of order n for a μ component link is oriented μ circles with n chords. A Jacobi diagram of order n for a μ component link is a trivalent graph with $2n$ vertices and it is a union of oriented μ circles and an internal graph G. When G is isomorphic to the standard n-tree, we call it a one branch tree diagram for a μ component link. If the number of components increases by deleting an edge from a graph, the edge is called *a cut edge*. In the case of links, a one branch tree diagram may have a cut edge.

Theorem 2.2([14]). *Let T be a one branch tree diagram of order n for a μ component link. If T has a cut edge, then the value of the Vassiliev invariant of order n for T is equal to* 0.

In the same way of the proof of Theorem 2.2, we can obtain Lemma 2.3.

Lemma 2.3. *Let T' be a Jacobi diagram of order n for a μ component link whose internal graph G is isomorphic to a union of standard n trees. If T' has a cut edge, then the value of the Vassiliev invariant of order n for T' is equal to* 0.

From now, we consider the coefficient of z^n in the Conway polynomial as a Vassiliev invariant of order n. Let $a_n(K)$ be the coefficient of the Conway polynomial for a knot K. T. Kanenobu and Y. Miyazawa([7]) gave a recursion formula to obtain the value $a_n(\alpha)$ for a chord digram of order n, α:

$$a_{n+1}(\alpha^{n+1}) = a_n(\alpha^n),$$

where α^{n+1} is a chord diagram of order $n+1$ and α^n is the chord diagram of order n which is obtained from α^{n+1} as shown in Fig. 9. And

$$a_0(L) = \begin{cases} 0 & r > 1 \\ 1 & r = 1 \end{cases},$$

where r is the number of component of a link L. From the above fomulas, if a chord diagram of order n has a separated circle, then the value of a_n for it is equal to 0.

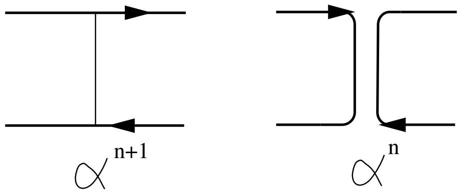


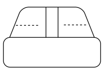
Fig. 9

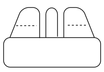
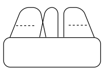
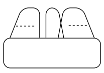
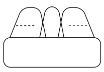
Lemma 2.4. *Let T_σ be a one branch tree diagram of order n. If there exist branches labelled i and $i+1$ $(2 \leq i \leq n-3)$ such that $\sigma(i)+1 = \sigma(i+1)$ in T_σ, then*

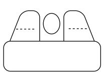
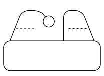
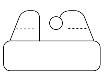
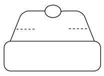
$$a_n(T_\sigma) = a_{n-2}(S),$$

where S is the one branch tree diagram of order $n-2$ which is obtained from T_σ by deleting two branches labelled i and $i+1$.

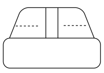
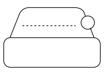
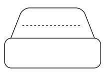
Proof. We consider the branches labelled i and $i+1$. By STU relations and Kanenobu and Miyazawa's recursion formula,

$a_n($  $)$

$= a_n($  $) - a_n($  $) - a_n($  $) + a_n($  $)$

$= a_{n-1}($  $) - a_{n-1}($  $) - a_{n-1}($  $) + a_{n-1}($  $).$

Since the first term has a separated circle, the value for it equals to 0. From Lemma 2.3, the values of the second and the third terms are also 0. The circle which apear in the last term can be slided to the branch 0 by IHX relations as is shown in Fig. 10. And by the recusion formula, we have

$a_n($  $) = a_{n-1}($  $) = a_{n-2}($  $).$

This completes the proof of Lemma 2.4. □

Here we consider a special kind of C_{n+1}-move $(n \geq 4)$ which is obtained from the standard C_n-move by changing the arc labelled i $(2 \leq i \leq n-2)$ to a hooked

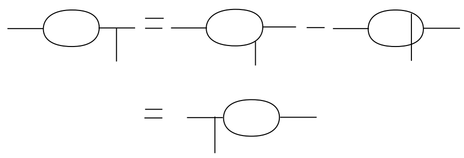


Fig. 10

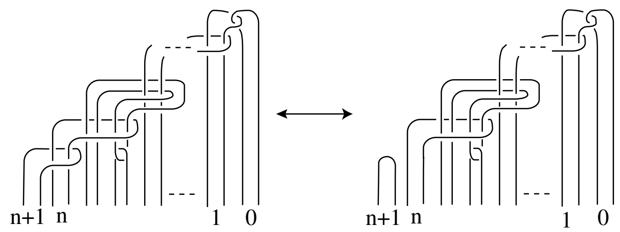


Fig. 11

arcs. We call it *a* C^i_{n+1}*-move*. The move in Fig. 11 is a C^{n-2}_{n+1}-move.

In [12], Nakanishi and the first author showed the similar result to Theorem 2.1 for the above C^{n-2}_{n+1}-move. By the same way of the proof for the result, we have Lemma 2.5.

Lemma 2.5. *If a knot K' is obtained from a knot K by a single C^i_{n+1}-move, then*

$$v_{n+1}(K') - v_{n+1}(K) = \pm v_{n+1}(T^i_\sigma),$$

where T^i_σ is the Jacobi diagram of order $n+1$ whose internal graph is isomorphic to the graph in Fig. 12 and $\sigma \in S_{n+1}$.

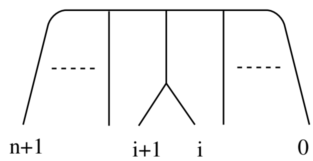


Fig. 12

In [12], it is also shown that a C^{n-2}_{n+1}-move cannot change the Conway polynomial: S. Naik and T. Stanford([11]) showed that two knots are S-equivalent if and only if they are transformed into each other by a finite sequence of double-delta moves in Fig. 13. Namely, a double-delta move cannot change the Conway polynomial. In [12], we show that the move depicted in Fig. 14 is realized by double-delta moves and the C_{n+1}^{n-2}-move is realized by double-delta moves.

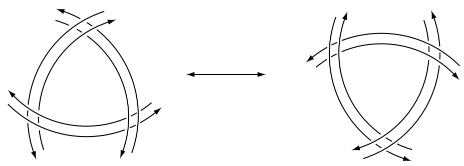


Fig. 13

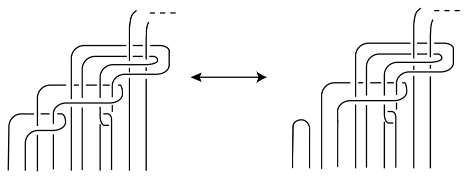


Fig. 14

For any C_{n+1}^{i}-move$(2 \leq i \leq n-2)$, we have Lemma 2.6.

Lemma 2.6. *If a knot K' is obtained from a knot K by a single C_{n+1}^{i}-move, then*

$$\nabla_K(z) = \nabla_{K'}(z),$$

where $\nabla_K(z)$ is the Conway polynomial for K.

Proof. By Fig. 15, a C_{n+1}^{i}-move is realized by the moves in Fig. 14. Then double-delta moves realize a C_{n+1}^{i}-move. □

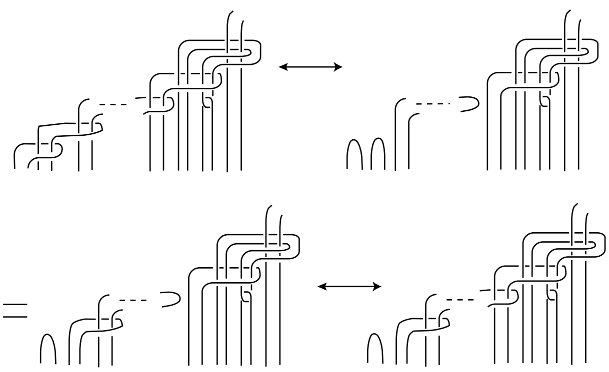


Fig. 15

From Lemmas 2.5 and 2.6, we have the following.

Corollary 2.7. *Let T^i_σ be the Jacobi diagram of order $n+1$ in Lemma 2.5. Then,*

$$a_{n+1}(T^i_\sigma) = 0.$$

Proof. Let K' be the knot which is obtained from a knot K by a C^i_{n+1}-move. Since the $n+1$th coefficient of the Conway polynomial a_{n+1} is a Vassiliev invariant of order $n+1$, by Lemma 2.5,

$$a_{n+1}(K') - a_{n+1}(K) = a_{n+1}(T^i_\sigma).$$

By Lemma 2.6,

$$a_{n+1}(K') = a_{n+1}(K).$$

Then we have

$$a_{n+1}(T^i_\sigma) = 0. \qquad \square$$

Lemma 2.8. *Let T_σ be a one branch tree diagram of order $2n$ $(n \geq 2)$ and $T'_{\sigma'}$ the one branch tree diagram which is obtained from T_σ by changing the branches labelled i and $i+1$ $(2 \leq i \leq 2n-3)$ as in Fig. 16. Then*

$$a_{2n}(T_\sigma) = a_{2n}(T'_{\sigma'}).$$

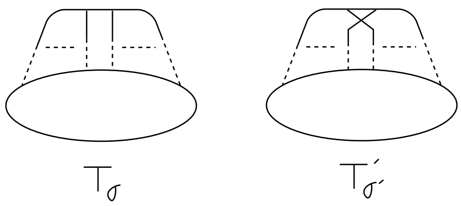


Fig. 16

Proof. By the IHX relation,

$$a_{2n}(T_\sigma) - a_{2n}(T'_{\sigma'}) = a_{2n}(T^i_\sigma),$$

where T^i_σ is the Jacobi diagram of order $2n$ in Lemma 2.5. By Corollary 2.7,

$$a_{2n}(T^i_\sigma) = 0.$$

Then we have

$$a_{2n}(T_\sigma) = a_{2n}(T'_{\sigma'}). \qquad \square$$

3. Proof of Theorem 1.3

Let K' be the knot which is obtained from K by a C_{2n}-move ($n \geq 2$). By Theorem 2.1,

$$a_{2n}(K') - a_{2n}(K) = \pm a_{2n}(T_\sigma), \tag{1}$$

where T_σ is a one branch tree diagram of order $2n$ and $\sigma \in S_{2n}$.

We can change an order of branches by using Lemma 2.8 and delete parallel branches by Lemma 2.4. From now we reduce T_σ to a simpler one branch tree diagram.

By changing an order of the branches labelled from 2 to $2n-2$ suitably, we can transform T_σ into the one branch tree T_τ such that $\tau \in S_{2n}$ satisfies

$$\tau(2n-2) > \tau(2n-3) > \cdots > \tau(3) > \tau(2).$$

And by Lemma 2.8,

$$a_{2n}(T_\sigma) = a_{2n}(T_\tau).$$

By the antisymmetry relation, we may assume that

$$\tau(2n) > \tau(2n-1).$$

For the order of $\tau(2n)$, $\tau(2n-1)$ and $\tau(1)$, there are the following cases;
Case (i) $\tau(2n-1) > \tau(1)$.
Case(ii) $\tau(2n) > \tau(1) > \tau(2n-1)$.
Case(iii) $\tau(1) > \tau(2n)$.
Changing the branches labelled 0 and 1 by the antisymmetry relation, Case (iii) is reduced to Case (i).

Case (i): The external circle in T_τ is divided into the four parts A, B, C and D by the four vertices numbered 0, $\tau(1)$, $\tau(2n-1)$ and $\tau(2n)$ as is shown in Fig. 17.

Since the order of T_τ is $2n$, we have $2n-3$ branches between the branches 1 and $2n-1$. If $n \geq 4$, the number of them is more than or equal to 5. Then at least one of four parts A, B, C and D has more than one ends of the branches. These branches are parallel. We delete two parallel branches by using Lemma 2.5, then we can reduce it to the case $n \leq 3$.

In the case $n = 2$, there exist the four cases in Fig. 18 for the positon of branches. Calculating the value of a_4 by using Kanenobu and Miyazawa's recursion formulas,

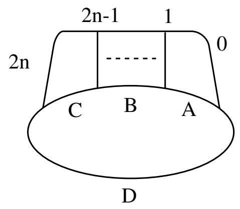


Fig. 17

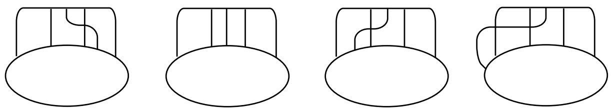


Fig. 18

we have that all of the values are equal to 0.

In the case $n = 3$, if one of four parts A, B, C and D has more than one ends of the branches, we can reduce it to the case $n = 2$. Then it is enoght to consider the four cases in Fig. 19. Calculating the values of a_6 for them, all of them are equal to 0.

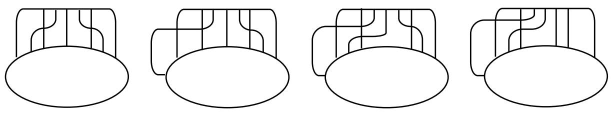


Fig. 19

Therefore, we have

$$a_{2n}(T_\sigma) = 0. \tag{2}$$

Case (ii): We consider the four parts A, B, C and D on the external circle as shown in Fig. 20.

We have $2n - 3$ branches between the branches 1 and $2n - 1$. If $n \geq 4$, we can reduce it to the case $n \leq 3$ in the same way of Case (i).

In the case $n = 2$, there are four cases in Fig. 21. Calculating the values of a_4 for them by the recusion formulas, we have that all of them eqaul to -2.

In the case $n = 3$, it is enough to consider four cases in Fig. 22 and the values of a_6 for them equal -2.

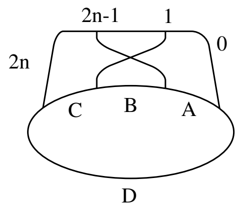


Fig. 20

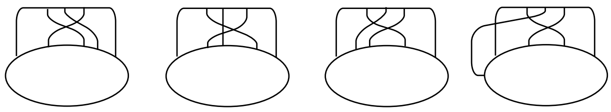


Fig. 21

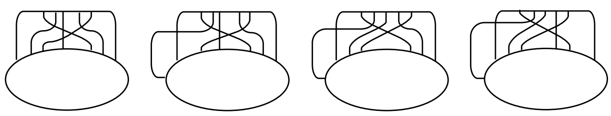


Fig. 22

Therefore, we have

$$a_{2n}(T_\sigma) = \pm 2. \tag{3}$$

From (1),(2) and (3),

$$a_{2n}(K') - a_{2n}(K) = 0, \pm 2.$$

This completes the proof of Theorem 1.3. □

4. Proof of Theorem 1.6

Let K_n be the knot in Fig. 23 and in [9] the Conway polynomial for K_n is calculated as the following:

$$\nabla_{K_n}(z) = 1 + \sum_{i=1}^{n-1} (-1)^i z^{2i}. \tag{4}$$

We consider the knot $K'_{2m,\alpha}$ in Fig. 24. Performing C_{2m}-moves on $K'_{2m,\alpha}$ α times, we have the knot K_{2m-1}.

From (4),

$$a_{2m}(K_{2m-1}) = (-1)^m.$$

The Jacobi diagram corresponding to the above C_{2m}-move is one in Case (ii) in the previous section. By the result of Case (ii), the antisymmetry relation and

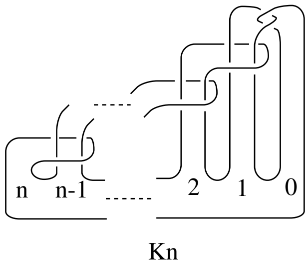


Fig. 23

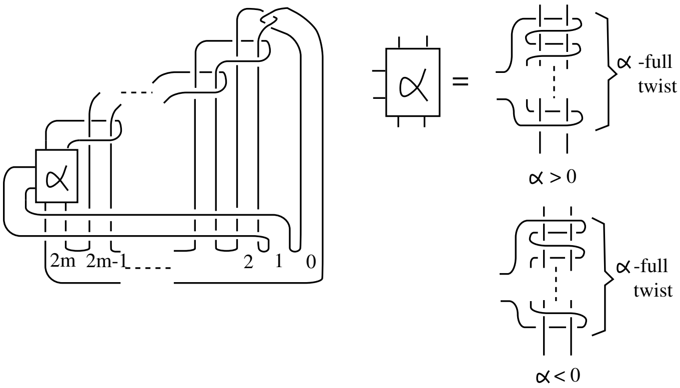


Fig. 24

Theorem 2.1 the above C_{2m}-move changes the value of a_{2m} by ± 2. By considering the orientations of the arcs in the C_{2m}-move, we can determine the sign. Then we have

$$a_{2m}(K'_{2m,\alpha}) = 2\alpha + (-1)^m.$$

Performing the C_n-move $(2n > 2m > n)$ on $K'_{2m,\alpha}$, we obtain K_{n-1}.

$$a_{2m}(K_{n-1}) = (-1)^m \text{or} \quad 0.$$

Therefore we have

$$a_{2m}(K'_{2m,\alpha}) - a_{2m}(K_{n-1}) = 2\alpha \quad \text{or} \quad 2\alpha + (-1)^m.$$

By Theorem 1.5 and the above argument, we have Theorem 1.6. □

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