KNOTS WITH GIVEN FINITE TYPE INVARIANTS AND CONWAY POLYNOMIAL

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ABSTRACT

It is well known that the coefficient of z^m of the Conway polynomial is a Vassiiev invariant of order m. In this paper, we show that for any given pair of a natural number n and a knot K, there exist infinitely many knots whose Vassiliev invariants of order less than or equal to n and Conway polynomials coincide with those of K.

Keywords: Vassiliev invariant, Conway polynomial, C_n-move

1. Introduction

Whenever we have a knot invariant v which takes value in some abelian group, we can define an invariant of singular knots by the Vassiliev skein relation:

$$v(K_D) = v(K_+) - v(K_-),$$

where *a singular knot* is an immersion of a circle into R^3 whose singularities are double points only and K_D, K_+ and K_- denote the diagrams of singular knots which are identical except near one point as shown in Fig. 1.

An invariant v is called *a Vassiliev invariant of order* n and denoted by v_n, if n is the smallest integer such that v vanishes on all singular knots with more than n double points. If a knot invariant is a Vassiliev invariant of order m for some integer m, we call it *an invariant of finite type.*

For any given integer n and any given knot K, some examples of knots have been constructed, all of whose Vassiliev invariants of order less than or equal to

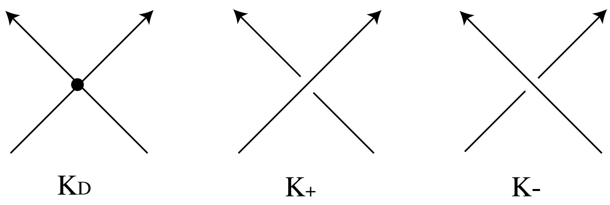


Fig. 1

n coincide with those of K([9],[15],[22]). And such examples of knots with more conditions are also constructed([13], [16], [18], [20], [25]).

A standard C_n-move is a local move depicted in Fig. 2 and *a C_1-move* is defined as a crossing change. M. N. Goussarov([4]) and K. Habiro([5], [6]) showed independently that two knots can be transformed into each other by a finite sequence of standard C_n-moves if and only if they have the same Vassiliev invariants of order less than n. C_n-moves are originally defined by Habiro in [5]. In [18] and [23], they are defined as a family of local moves. It is known that any kind of C_n-moves can be realized by a finite sequence of standard C_n-moves.

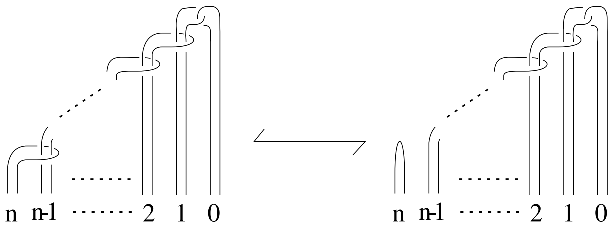


Fig. 2

In this paper, we show the following theorem by making use of a special kind of C_n-moves.

Main Theorem. *Let n be a natural number and K a knot. Then there are infinitely many knots J_m ($m = 1, 2, \ldots$) which satisfy the following:*
(1) *for any v_k ($k = 1, 2, \ldots, n$), $v_k(J_m) = v_k(K)$, and*
(2) *$\nabla_{J_m}(z) = \nabla_K(z)$, where $\nabla_K(z)$ is the Conway polynomial of K.*

2. C_n-moves and Jacobi diagrams

A tangle T is a disjoint union of properly embedded arcs in the unit 3-ball B^3. A tangle T is *trivial* if there exisits a properly embedded disk in B^3 containing T. *A local move* is a pair of trivial tangles (T_1, T_2) with $\partial T_1 = \partial T_2$ such that for each component t of T_1 there exists a component u of T_2 with $\partial t = \partial u$. Let

(T_1, T_2) be a local move, t_1 a component of T_1 and t_2 a component of T_2 such that $\partial t_1 = \partial t_2$. Replacing t_1 and t_2 by hooked arcs in Fig. 3, a new kind of local move can be obtained. This local move is called a *double of* (T_1, T_2) *with respect to the components* t_1 *and* t_2.

A C_1*-move* is a local move as illustrated in Fig. 4. A double of a C_k-move is called *a* C_{k+1}*-move.* For details, refer to [18] or [23].

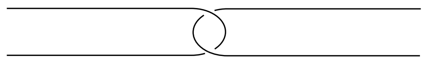


Fig. 3

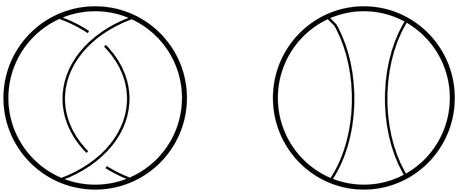


Fig. 4

By K^m, we denote a singular knot with m double points. From the definition of the Vassiiev invariant, $v_m(K^m)$ does not change by a crossing change and it is determined by the position of double points on K^m. To describe the position of double points, the notion of a chord diagram is introduced in [3]. *A chord diagram of order* n is a counterclockwise oriented circle with n chords. We may associate it to a singular knot with n double points by connecting the preimage of each double point with a chord. The value of a Vassiliev invariant for a chord diagram is defined as the value for a singular knot that is associated with the chord diagram.

Chord diagrams are generalized to Jacobi diagrams in [1]. *A Jacobi diagram of order* n is a trivalent graph with $2n$ vertices. It is a union of a circle and an internal graph G. The circle is oriented and the other edges are all unoriented. Each trivalent vertex on G has an orientation, that is a cyclic ordering of the edges incident to it.

Let A_n be the additive group generated by the chord diagrams of order n modulo 4T relations as is shown in Fig. 5 and B_n the additive group generated by the Jacobi diagrams of order n modulo STU relations in Fig. 6. Then the isomorphism between A_n and B_n is induced by the inclusion of chord diagrams into Jacobi diagrams([1]).

A one branch tree diagram T is a special kind of Jacobi diagrams whose internal graph G is isomorphic to a standard n-tree in Fig. 7 preserving the vertex

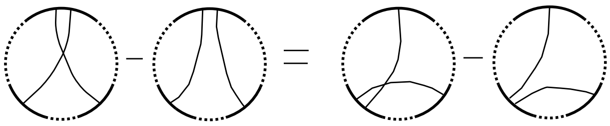


Fig. 5

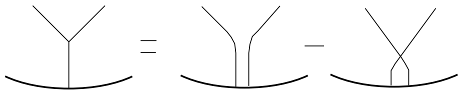


Fig. 6

orientations([14]). Label the branches of the standard n-tree as in Fig. 7. The branches of G may be labelled under the isomorophism between G and the standard n-tree. And number the vertices of the circle of T by 0, 1, 2, ..., n in the counterclockwise direction such that the end the branch 0 of G is numbered by 0. The correspondence between the labels of branches of G and the numbers of their end points on the circle determines a permutation $\sigma \in S_n$. Conversely, if a permutation $\sigma \in S_n$ is given, a unique one branch tree diagram T can be constructed. Then we denote a one branch tree diagram by T_σ. By STU-relations, a one branch tree diagram can be expressed as a signed sum of chord diagrams. The value of a Vassiliev invariant for a one branch tree diagram is defined as the signed sum of the value for chord diagrams.

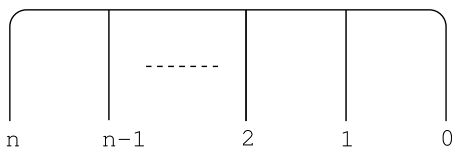


Fig. 7

A one branch tree diagram is closely related to a standard C_n-move.

Theorem 2.1([19]). *Let v_n be a Vassiliev invariant of order n. If a knot L is obtained from a knot K by a single standard C_n-move, then*

$$v_n(K) - v_n(L) = \pm v_n(T_\sigma),$$

where T_σ is a one branch tree diagram of order n.

In Theorem 2.1, the one branch tree diagram T_σ is determined by the position of arcs in the C_n-move on a knot K and the sign of the formula depend only on the orientations of arcs in the C_n-move. By the same way of the proof of Theorem 2.1, for a singular knot with m double points K^m, we have the following.

Theorem 2.1'. *If a singular knot L^m is obtained from a singular knot K^m by a single standard C_n-move, then*

$$v_{m+n}(K^m) - v_{m+n}(L^m) = \pm v_{m+n}(T_\sigma),$$

where T_σ is a Jacobi diagram of order $m+n$ whose internal graph G is isomorphic to the union of m chords and a one branch tree diagram of order n.

Here, we consider the C_{n+1}-move ($n \geq 4$) as is shown in Fig. 8 which is obtained from the standard C_n-move in Fig. 2 by changing the arc labelled $n-2$ into the hooked arcs in Fig. 3. We denote it by C'_{n+1}-move. A C'_{n+1}-move is also related to some kind of Jacobi diagrams.

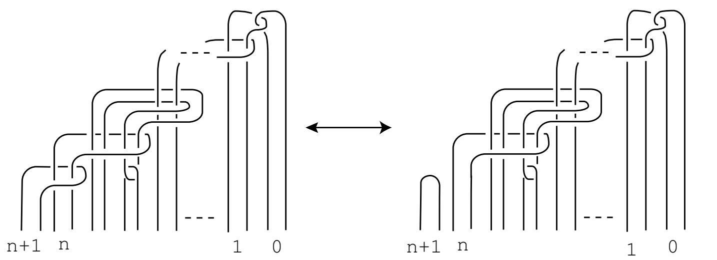


Fig. 8

Lemma 2.2. *If a knot L is obtained from a knot K by a single C'_{n+1}-move, then*

$$v_{n+1}(K) - v_{n+1}(L) = \pm v_{n+1}(T'_\sigma),$$

where T'_σ is a one branch tree diagram whose internal graph G is isomorphic to the graph in Fig. 9 and $\sigma \in S_{n+1}$.

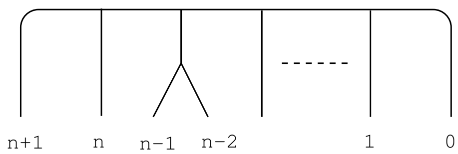


Fig. 9

Proof. Let c be a crossing of hooked arcs in the C'_{n+1}-move in Fig. 8 and ε the sign of c. By $K(\bar{c})$ and $K(\dot{c})$, we denote the knot which is obtained from K by changing the crossing at c and the singular knot which is obtained from K by making the crossing into the double point at c, respectively. From the definition of a Vassiliev invariant,

$$\begin{gathered} v_{n+1}(K) - v_{n+1}(L) \\ = v_{n+1}(K(\bar{c})) + \varepsilon\ v_{n+1}(K(\dot{c})) - \{v_{n+1}(L(\bar{c})) + \varepsilon\ v_{n+1}(L(\dot{c}))\}. \end{gathered}$$

Since two knot $K(\bar{c})$ and $L(\bar{c})$ are the same knot type, then

$$v_{n+1}(K) - v_{n+1}(L) = \varepsilon\{v_{n+1}(K(\dot{c})) - v_{n+1}(L(\dot{c}))\}. \qquad (1)$$

Let $K'(\dot{c})$ be the singular knot that is obtained from $K(\dot{c})$ by a standard C_n-move as shown in Fig. 10. It is noted that we have the singular knot $L(\dot{c})$ from $K'(\dot{c})$ by a standard C_n-move.

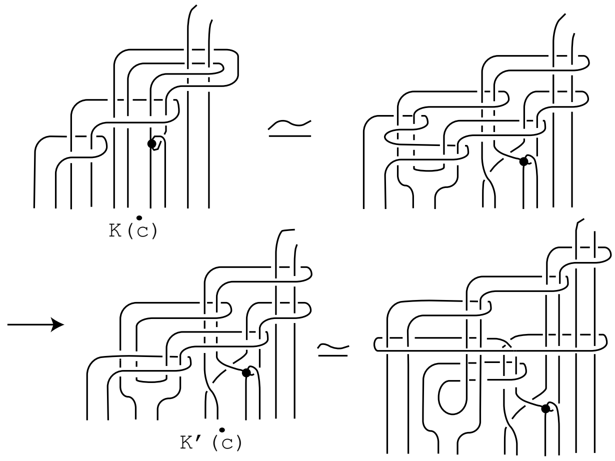


Fig. 10

By Theorem 2.1', we have

$$\begin{gathered} v_{n+1}(K(\dot{c})) - v_{n+1}(K'(\dot{c})) = \pm v_{n+1}(T) \\ v_{n+1}(K'(\dot{c})) - v_{n+1}(L(\dot{c})) = \mp v_{n+1}(T'). \end{gathered}$$

On the above C_n-moves, the orientation of only one arc is different. The sign of the right side of above formulas is opposite. By the proof of Theorem 2.1 and by considering the position of the crossing c, the Jacobi diagram T is one of two Jacobi diagrams in Fig. 11 and T' is the other. From the above formulas,

$$\begin{aligned} & v_{n+1}(K(\dot{c})) - v_{n+1}(L(\dot{c})) \\ = \ & v_{n+1}(K(\dot{c})) - v_{n+1}(K'(\dot{c})) + v_{n+1}(K'(\dot{c})) - v_{n+1}(L(\dot{c})) \\ = \ & \pm\{v_{n+1}(T) - v_{n+1}(T')\}. \qquad (2) \end{aligned}$$

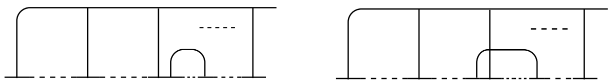


Fig. 11

By the STU-relation, we have

$$v_{n+1}(T) - v_{n+1}(T') = \pm v_{n+1}(T'_\sigma). \tag{3}$$

From (1), (2) and (3),

$$v_{n+1}(K) - v_{n+1}(L) = \pm v_{n+1}(T'_\sigma). \qquad \square$$

We can define a Vassiiev invariant for μ component links as for knots. A singular μ component link is an immersion of μ circles into R^3. And we can also define a chord diagram and a Jacobi diagram for links. A chord diagram of order n for a μ component link is oriented μ circles with n chords. A Jacobi diagram of order n for a μ component link is a trivalent graph with $2n$ vertices and it is a union of oriented μ circles and the graph G. If the number of components increases by deleting an edge from a graph, the edge is called *a cut edge*. In the case of links, a one branch tree diagram may have a cut edge.

Theorem 2.3([17]). *Let T be a one branch tree diagram of order n for a μ component link. If T has a cut edge, then the value of the Vassiliev invariant of order n for T is equal to* 0.

3. Proof of Main Theorem

In this section, we will prove Main theorem by using C'_{n+1}-moves.

Lemma 3.1. *If a knot L is obtained from a knot K by C'_{n+1}-moves, then*

$$\nabla_L(z) = \nabla_K(z),$$

where $\nabla_K(z)$ is the Conway polynomial of a knot K.

Proof. A local move in Fig. 12 is called a doubled-delta move. S. Naik and T. Stanford showed that two knots are S-equivalent if and only if they are transformed into each other by a finite sequence of doubled-delta moves in [12]. The local move in Fig. 13 is the same move as a doubled-delta move and the C'_{n+1}-move is generated by the move in Fig. 13 as is shown in Fig. 14. Then the Conway polynomial cannnot be changed by C'_{n+1}-moves. $\square$

Let $V^{(n)}(L;1)$ be the n-th derivative of the Jones polynomial $V(L;t)$ ([7]) of a link L evaluated at 1. It is a Vassiiev invariant of order n. In [8], T. Kanenobu and

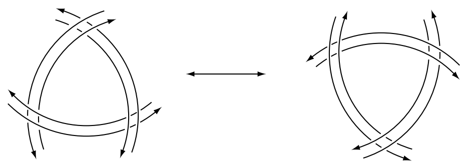


Fig. 12

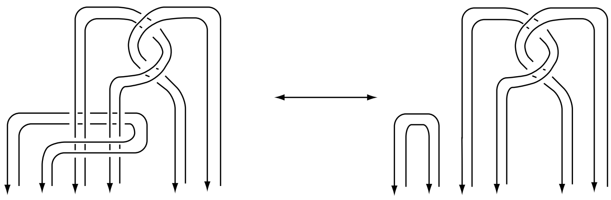


Fig. 13

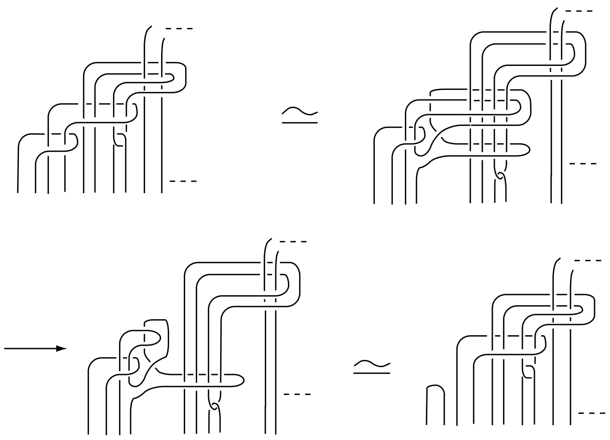


Fig. 14

Y. Miyazawa gave a recursion formula to obtain the value of $V^{(n)}(\alpha;1)$ for a chord diagram of order n, α :

$$V^{(n+1)}(\alpha^{n+1}) = 2(n+1)V^{(n)}(\alpha^n_-) + (n+1)V^{(n)}(\alpha^n_0),$$

where α^{n+1} is a chord diagram of order $n+1$ and α^n_- and α^n_0 are chord diagrams of order n which is obtained from α^{n+1} as depicted in Fig. 15. And

$$V(L;1) = (-2)^{r-1},$$

where r is the number of components of a link L.

Now we consider the Jacobi diagram T'_{id} that is the Jacobi diagram in Lemma 2.2 in the case the permutation $\sigma \in S_{n+1}$ is the identity.

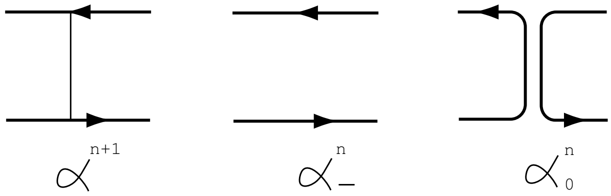
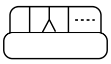
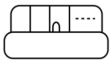
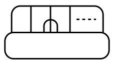


Fig. 15

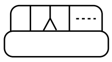
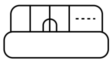
Lemma 3.2. *For the Jacobi diagram T'_{id}, it holds the following:*

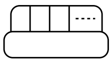
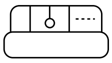
$$V^{(n+1)}(T'_{id}) = 3(-2)^{n-1}(n+1)!.$$

Proof. By the STU-relation,

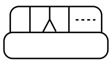
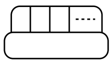
$V^{(n+1)}($  $) = V^{(n+1)}($  $) - V^{(n+1)}($  $).$

If a chord diagram has a isolate chord which does not have intersection with other chords, the value of the Vassiiev invariant for it equals to 0. And by the recursion formula,

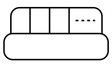
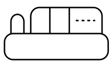
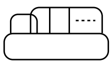
$V^{(n+1)}($  $) = -V^{(n+1)}($  $)$

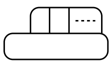
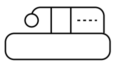
$= -\{2(n+1)V^{(n)}($  $) + (n+1)V^{(n)}($  $)\}.$

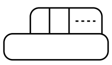
By Theorem 2.3, we have

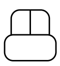
$V^{(n+1)}($  $) = -2(n+1)V^{(n)}($  $)$ (4)

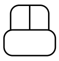
Similarly, by using the STU-relation, the recursion formula and Theorem 2.3,

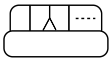
$V^{(n)}($  $) \;=\; V^{(n)}($  $) - V^{(n)}($  $)$

$= -\{2n\, V^{(n-1)}($  $) + n\, V^{(n-1)}($  $)\}$

$= -2n\, V^{(n-1)}($  $)$

$= (-2)n(-2)(n-1)\cdots(-2)3\, V^{(2)}($  $).$ (5)

By using the recursion formula, we have $V^{(2)}($  $) = 6$. From (4) and (5), it follows

$V^{(n+1)}($  $) = 3(-2)^{n-1}(n+1)!.$

This completes the proof of Lemma 3.2. □

Proof of Main Theorem.

We consider the C'_{n+1}-move such that the labels of arcs appear in numerical order along the orientation of the knot. By J_m, we denote the knot that is obtained from a given knot K by performing the C'_{n+1}-moves m times.

Any kind of C_{n+1}-moves cannot change the Vassiliev invariants of order less than or eqaul to n. Then

$$v_k(K) = v_k(J_m) \ (k = 1, 2, \ldots, n).$$

By Lemma 3.1,

$$\nabla_K(z) = \nabla_{J_m}(z).$$

In Lemma 2.2, the sign of the formula is determined only by the orientations of arcs in the C'_{n+1}-move as in Theorem 2.1. Since we repeat the same C'_{n+1}-move on the knot K, then

$$V^{(n+1)}(J_m; 1) - V^{(n+1)}(J_{m+1}; 1) = V^{(n+1)}(J_{m+1}; 1) - V^{(n+1)}(J_{m+2}; 1).$$

By Lemma 3.2,

$$V^{(n+1)}(J_m; 1) \neq V^{(n+1)}(J_\ell; 1) \ (m \neq \ell).$$

Then we have Main Theorem.

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