Almost Alternating Knots Producing an Alternating Knot

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Abstract

Adams et al. introduce the notion of almost alternating links; non-alternating links which have a projection whose one crossing change yields an alternating projection. For an alternating knot $K$, we consider the number $\text{Alm}(K)$ of almost alternating knots which have a projection whose one crossing change yields $K$. We show that for any given natural number $n$, there is an alternating knot $K$ with $\text{Alm}(K) \geq n$.

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1. Introduction

The notion of almost alternating links is introduced by Adams et al. [2]. A projection of a link $L$ is almost alternating if one crossing change makes the projection alternating. The crossing point on the almost alternating projection which produces an alternating projection is called the dealternator. A link $L$ is almost alternating if $L$ has an almost alternating projection and does not have an alternating projection. We note that an almost alternating link has infinitely many almost alternating projections by using the move at a dealternator in Fig. 1 repeatedly. Then for an almost alternating knot $L$, there are infinitely many alternating knots which guarantee that $L$ is an almost alternating.

Conversely, for an alternating knot $K$, we consider an almost alternating knot $L$ which has a projection whose one crossing change produces $K$. In the case there exists an almost alternating knot $L$ producing an alternating knot $K$, if we change
the crossing corresponding to the dealternator on an alternating projection of $K$, we have a projection of $L$.

For an alternating knot $K$, by $\text{Alm}(K)$, we denote the number of almost alternating knots which have a projection whose one crossing change yields $K$.

Since the knots whose minimum crossing numbers are less than or equal to 7 are alternating, we have Proposition 1.1.

**Proposition 1.1.** Let $c(K)$ be the minimum crossing number of a knot $K$. If $K$ is an alternating knot with $c(K) \leq 7$, then $\text{Alm}(K) = 0$.

In this paper, we show the following:

**Theorem 1.2.** For any given natural number $n$, there is an alternating knot $K$ with $\text{Alm}(K) \geq n$.

2. **Proof of Theorem 1.2**

Let $L_i$ be an alternating knot as is shown in Fig. 2 and $L'_i$ the knot which is obtained from $L_i$ by changing the crossing at $c_i$ ($i = 1, 2, \ldots, n$). Let $K = L_1 \sharp L_2 \sharp \cdots \sharp L_n$ and $K_i = L_1 \sharp L_2 \sharp \cdots \sharp L'_i \sharp \cdots \sharp L_n$ ($i = 1, 2, \ldots, n$). Then, $K$ is an alternating knot and $K_i$ has an almost alternating projection whose one crossing change yields the alternating projection of $K$.
By spanning a Seifert surface according to the Seifert algorithm, we have the following $(2^i + 4) \times (2^i + 4)$ Seifert matrix $M_i$ for $L_i'$.

$$M_i = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 \\
\end{pmatrix}.$$

Then, we have

$$\det(M_i - tM_i^T) = \begin{vmatrix}
-1 + t & 1 & 0 & -t & -t & 0 \\
-t & -1 + t & 1 & 0 & 0 & 0 \\
0 & -t & -1 + t & 0 & 0 & 0 \\
1 & 0 & 0 & -1 + t & 0 & 0 \\
1 & 0 & 0 & 0 & -1 + t & -t \\
0 & 0 & 0 & 0 & 1 & -1 + t \\
0 & 0 & 0 & 0 & 1 & -1 + t \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-t & 0 & 0 & 0 & 0 & 0 \\
-1 + t & -t & 0 & 0 & 0 & 0 \\
1 & -1 + t & -t & 0 & 0 & 0 \\
0 & 1 & -1 + t & -t & 0 & 0 \\
0 & 0 & 1 & -1 + t & -t & 0 \\
0 & 0 & 0 & 1 & -1 + t & 0 \\
0 & 0 & 0 & 0 & 1 & -1 + t \\
\end{vmatrix}.$$

Let $\Delta_{L_i'}$ be the Alexander polynomial of $L_i'$. The following formulas are obtained:

$$\Delta_{L_1'} = (t^2 + 1)\Delta_{L_{i-1}} - t^2\Delta_{L_{i-2}}.$$

$$\Delta_{L_2'} = (t^5 - 1)(t - 1) + t^3.$$

$$\Delta_{L_2'} = (t^7 - 1)(t - 1) + t^3(t^2 - t + 1).$$

By induction, it follows that

$$\Delta_{L_i'} = (t^{2i+3} - 1)(t - 1) + t^3\sum_{k=0}^{2i-2} (-t)^k$$

$$= t^{2i+4} - t^{2i+3} + t^{2i+1} - t^{2i} + \cdots + t^3 - t + 1. \quad (2.1)$$


**Theorem 2.1 [4].** For an alternating knot $K$, all coefficients from the lowest degree to the highest degree of $\Delta_K$ are non-zero.
From (2.1), the coefficients of $t^{2i+2}$ and $t^2$ of $\Delta L_i'$ are zero. Then we have Lemma 2.2.

**Theorem 2.2.** The knot $L'_i$ ($i = 1, 2, \ldots, n$) is non-alternating.

Let $P$ be the projection plane on which the projection $\tilde{L}$ of a link $L$ exists. Menasco [3] shows Theorem 2.3.

**Theorem 2.3** [3]. Let $L$ be a non-split alternating link. For each disc $D$ on the projection plane $P$ with $\partial D$ meeting an alternating projection $\tilde{L}$ in just two points, if $\tilde{L} \cap D$ is an embedded arc, $L$ is prime.

By using Lemma 2.2 and Theorem 2.3, we have Lemma 2.4.

**Lemma 2.4.** The knot $K_i = L_1 \sharp L_2 \sharp \cdots \sharp L'_i \sharp \cdots \sharp L_n$ ($i = 1, 2, \ldots, n$) is non-alternating.

**Proof.** By Theorem 2.3, if $K_i$ is a non-prime alternating knot, then there is a disc $D$ with $\partial D$ meeting an alternating projection $\tilde{K}_i$ in just two points such that the interior and the exterior of $D$ represent factor knots. And these factor knots are alternating. By Lemma 2.2, $L'_i$ is non-alternating. Therefore, $K_i = L_1 \sharp L_2 \sharp \cdots \sharp L'_i \sharp \cdots \sharp L_n$ is non-alternating.

**Lemma 2.5.** The knot types $K_i = L_1 \sharp L_2 \sharp \cdots \sharp L'_i \sharp \cdots \sharp L_n$ and $K_j = L_1 \sharp L_2 \sharp \cdots \sharp L'_j \sharp \cdots \sharp L_n$ ($i < j, i, j = 1, 2, \ldots, n$) are different.

**Proof.** The knot $K_i (i = 1, 2, \ldots, j - 1)$ has the alternating knot $L_j$ with minimum crossing number $2j + 7$ as a factor knot. However, $K_j$ does not have $L_j$ as a factor knot. Therefore, $K_i$ and $K_j$ are different knot types. Since it holds for any $j (j = 2, 3, \ldots, n)$, we have Lemma 2.5.

By Lemma 2.4, each $K_i = L_1 \sharp L_2 \sharp \cdots \sharp L'_i \sharp \cdots \sharp L_n$ ($i = 1, 2, \ldots, n$) is an almost alternating knot whose one crossing change yields $K = L_1 \sharp L_2 \sharp \cdots \sharp L_n$. By Lemma 2.5, $K_i$ and $K_j$ ($i \neq j$) represent different knot types. This completes the proof of Theorem 1.2.

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**References**

