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ALMOST ALTERNATING KNOTS PRODUCING AN ALTERNATING KNOT

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ABSTRACT

Adams *et al.* introduce the notion of almost alternating links; non-alternating links which have a projection whose one crossing change yields an alternating projection. For an alternating knot K, we consider the number $\mathrm{Alm}(K)$ of almost alternating knots which have a projection whose one crossing change yields K. We show that for any given natural number n, there is an alternating knot K with $\mathrm{Alm}(K) \geq n$.

Keywords: Almost alternating knot; alternating knot.

Mathematics Subject Classification 2010: 57M25

1. Introduction

The notion of almost alternating links is introduced by Adams *et al.* [2]. A projection of a link L is *almost alternating* if one crossing change makes the projection alternating. The crossing point on the almost alternating projection which produces an alternating projection is called the *dealternator*. A link L is *almost alternating* if L has an almost alternating projection and does not have an alternating projection. We note that an almost alternating link has infinitely many almost alternating projections by using the move at a dealternator in Fig. 1 repeatedly. Then for an almost alternating knot L, there are infinitely many alternating knots which guarantee that L is an almost alternating.

Conversely, for an alternating knot K, we consider an almost alternating knot L which has a projection whose one crossing change produces K. In the case there exists an almost alternating knot L producing an alternating knot K, if we change

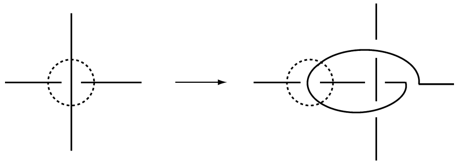

Fig. 1.

the crossing corresponding to the dealternator on an alternating projection of K, we have a projection of L.

For an alternating knot K, by $\mathrm{Alm}(K)$, we denote the number of almost alternating knots which have a projection whose one crossing change yields K.

Since the knots whose minimum crossing numbers are less than or equal to 7 are alternating, we have Proposition 1.1.

Proposition 1.1. *Let $c(K)$ be the minimum crossing number of a knot K. If K is an alternating knot with $c(K) \leq 7$, then* $\mathrm{Alm}(K) = 0$.

In this paper, we show the following:

Theorem 1.2. *For any given natural number n, there is an alternating knot K with* $\mathrm{Alm}(K) \geq n$.

2. Proof of Theorem 1.2

Let L_i be an alternating knot as is shown in Fig. 2 and L'_i the knot which is obtained from L_i by changing the crossing at c_i $(i = 1, 2, \ldots, n)$. Let $K = L_1 \sharp L_2 \sharp \cdots \sharp L_n$ and $K_i = L_1 \sharp L_2 \sharp \cdots \sharp L'_i \sharp \cdots \sharp L_n$ $(i = 1, 2, \ldots, n)$. Then, K is an alternating knot and K_i has an almost alternating projection whose one crossing change yields the alternating projection of K.

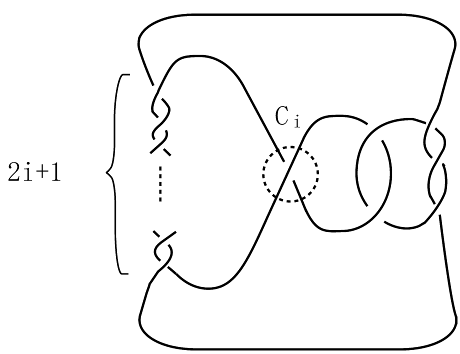

Fig. 2.

By spanning a Seifert surface according to the Seifert algorithm, we have the following $(2i+4)\times(2i+4)$ Seifert matrix M_i for L_i'.

$$
M_i = \left(\begin{array}{cccccccccccc}
-1 & 1 & 0 & 0 & 0 & 0 & & & & & & \\
0 & -1 & 1 & 0 & 0 & 0 & & & & & & \\
0 & 0 & -1 & 0 & 0 & 0 & & & & & & \\
1 & 0 & 0 & -1 & 0 & 0 & & & & \text{\Large 0} & & \\
1 & 0 & 0 & 0 & -1 & 0 & & & & & & \\
0 & 0 & 0 & 0 & 1 & -1 & & & & & & \\
 & & & & & & \cdots & & & & & \\
 & & & & & & & 1 & -1 & 0 & 0 & 0 \\
 & & & \text{\Large 0} & & & & 0 & 1 & -1 & 0 & 0 \\
 & & & & & & & 0 & 0 & 1 & -1 & 0 \\
 & & & & & & & 0 & 0 & 0 & 1 & -1
\end{array}\right).
$$

Then, we have

$$
\det(M_i - tM_i^T)
$$

$$
= \left|\begin{array}{ccccccccccc}
-1+t & 1 & 0 & -t & -t & 0 & & & & & \\
-t & -1+t & 1 & 0 & 0 & 0 & & & & & \\
0 & -t & -1+t & 0 & 0 & 0 & & & & & \\
1 & 0 & 0 & -1+t & 0 & 0 & & & \text{\Large 0} & & \\
1 & 0 & 0 & 0 & -1+t & -t & & & & & \\
0 & 0 & 0 & 0 & 1 & -1+t & & & & & \\
 & & & & & & \cdots & & & & \\
 & & & & & & & -t & 0 & 0 & 0 \\
 & & & & & & & -1+t & -t & 0 & 0 \\
 & & & \text{\Large 0} & & & & 1 & -1+t & -t & 0 \\
 & & & & & & & 0 & 1 & -1+t & -t \\
 & & & & & & & 0 & 0 & 1 & -1+t
\end{array}\right|.
$$

Let $\Delta_{L_i'}$ be the Alexander polynomial of L_i'. The following formulas are obtained:

$$
\begin{aligned}
\Delta_{L_i'} &= (t^2+1)\Delta_{L_{i-1}'} - t^2\Delta_{L_{i-2}'}.\\
\Delta_{L_1'} &= (t^5-1)(t-1)+t^3.\\
\Delta_{L_2'} &= (t^7-1)(t-1)+t^3(t^2-t+1).
\end{aligned}
$$

By induction, it follows that

$$
\begin{aligned}
\Delta_{L_i'} &= (t^{2i+3}-1)(t-1) + t^3\sum_{k=0}^{2i-2}(-t)^k \\
&= t^{2i+4} - t^{2i+3} + t^{2i+1} - t^{2i} + \cdots + t^3 - t + 1. \qquad (2.1)
\end{aligned}
$$

Murasugi [4] characterized the Alexander polynomials for alternating knots.

Theorem 2.1 [4]. *For an alternating knot K, all coefficients from the lowest degree to the highest degree of Δ_K are non-zero.*

From (2.1), the coefficients of t^{2i+2} and t^2 of $\Delta_{L'_i}$ are zero. Then we have Lemma 2.2.

Theorem 2.2. *The knot* L'_i $(i = 1, 2, \dots, n)$ *is non-alternating.*

Let P be the projection plane on which the projection $\tilde{L}$ of a link L exists. Menasco [3] shows Theorem 2.3.

Theorem 2.3 [3]. *Let* L *be a non-split alternating link. For each disc* D *on the projection plane* P *with* ∂D *meeting an alternating projection* $\tilde{L}$ *in just two points, if* $\tilde{L} \cap D$ *is an embedded arc,* L *is prime.*

By using Lemma 2.2 and Theorem 2.3, we have Lemma 2.4.

Lemma 2.4. *The knot* $K_i = L_1 \sharp L_2 \sharp \cdots \sharp L'_i \sharp \cdots \sharp L_n$ $(i = 1, 2, \dots, n)$ *is non-alternating.*

Proof. By Theorem 2.3, if K_i is a non-prime alternating knot, then there is a disc D with ∂D meeting an alternating projection $\tilde{K_i}$ in just two points such that the interior and the exterior of D represent factor knots. And these factor knots are alternating. By Lemma 2.2, L'_i is non-alternating. Therefore, $K_i = L_1 \sharp L_2 \sharp \cdots \sharp L'_i \sharp \cdots \sharp L_n$ is non-alternating. □

Lemma 2.5. *The knot types* $K_i = L_1 \sharp L_2 \sharp \cdots \sharp L'_i \sharp \cdots \sharp L_n$ *and* $K_j = L_1 \sharp L_2 \sharp \cdots \sharp L'_j \sharp \cdots \sharp L_n$ $(i < j, i, j = 1, 2, \dots, n)$ *are different.*

Proof. The knot $K_i (i = 1, 2, \dots, j-1)$ has the alternating knot L_j with minimum crossing number $2j + 7$ as a factor knot. However, K_j does not have L_j as a factor knot. Therefore, K_i and K_j are different knot types. Since it holds for any $j (j = 2, 3, \dots, n)$, we have Lemma 2.5. □

By Lemma 2.4, each $K_i = L_1 \sharp L_2 \sharp \cdots \sharp L'_i \sharp \cdots \sharp L_n$ $(i = 1, 2, \dots, n)$ is an almost alternating knot whose one crossing change yields $K = L_1 \sharp L_2 \sharp \cdots \sharp L_n$. By Lemma 2.5, K_i and K_j $(i \neq j)$ represent different knot types. This completes the proof of Theorem 1.2.

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