THE C_k-GORDIAN COMPLEX OF KNOTS

YOSHIYUKI OHYAMA

Department of Mathematics, College of Arts and Sciences
Tokyo Woman's Christian University
2-6-1, Zempukuji, Suginami-ku, Tokyo, 167-8585, Japan
E-mail address: ohyama@lab.twcu.ac.jp

ABSTRACT

M.Hirasawa and Y.Uchida defined the Gordian complex of knots which is a simplicial complex whose vertices consist of all knot types in S^3 by using "a crossing change". In this paper, we define the C_k-Gordian complex of knots which is an extension of the Gordian complex of knots. Let k be a natural number more than 2 and we show that for any knot K_0 and any given natural number n, there exists a family of knots $\{K_0, K_1, \ldots, K_n\}$ such that for any pair (K_i, K_j) of distinct elements of the family, the C_k-distance $d_{C_k}(K_i, K_j) = 1$.

Keywords: Vassiliev invariant, Gordian distance, C_k-move, C_k-distance

1. Introduction

When we have a knot invariant v which takes value in some abelian group, we can extend it to an invariant of singular knots by the following:

$$v(K_D) = v(K_+) - v(K_-),$$

where *a singular knot* is an immersion of a circle into R^3 whose singularities are double points only and K_D, K_+ and K_- denote the diagrams of singular knots which are identical except near one point as shown in Fig. 1.1.

An invariant v is called *a Vassiliev invariant of order* n and denoted by v_n, if n is the smallest integer such that v vanishes on all singular knots with more than n double points.

A C_k*-move* is a local move depicted in Fig. 1.2 and *a* C_1*-move* is defined as a crossing change. M. N. Goussarov([4]) and K. Habiro([5], [6]) showed independently that two knots can be transformed into each other by a finite sequence of C_k-moves if and only if they have the same Vassiliev invariants of order less than k. If a knot K can be transformed into a knot K' by C_k-moves, we denote the minimum number of times of C_k-moves needed to transform K into K' by $d_{C_k}(K, K')$ and call it the *C_k-distance between K and K'*. The C_1-distance is usually called the Gordian distance and denoted by $d_G(K, K')$ since a C_1-move is a crossing change.

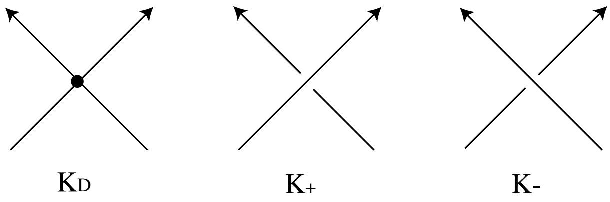

Fig.1.1

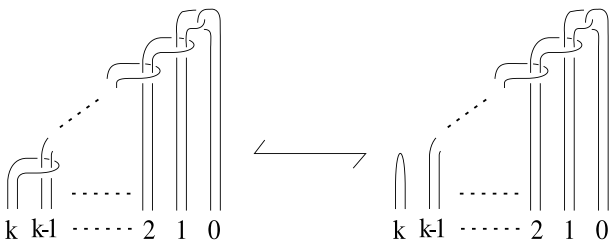

Fig.1.2

Some results concerning Vassiliev invariants and C_k-distances have been published ([13], [14], [16], [19]) and Y. Nakanishi and the author showed the following theorem in [12].

Theorem1.1([12]). *Let n be a natural number and K a knot. Then there are infinitely many knots J_m $(m = 1, 2, \ldots)$ which satisfy the following:*
(1) *for any v_k $(k = 1, 2, \ldots, n)$, $v_k(J_m) = v_k(K)$, and*
(2) $d_{C_k}(J_m, K) = \begin{cases} 1 & (k \neq 2, k = 1, \ldots, n) \\ 2 & (k = 2) \end{cases}$.

Here, it arises the problem whether or not we can add the condition that $d_{C_k}(J_m, J_\ell) = 1$ for $m \neq \ell$ in Theorem 1.1. Recently M. Hirasawa and Y. Uchida([7]) defined the simplicial complex called the Gordian complex of knots whose vertices consist of all knot types. They defined that a family of $n + 1$ vertices spans an n-simplex if the Gordian distance is equal to 1 for any pair of distinct elements of the family. We extend the notion, the Gordian complex of knots, to the C_k-Gordian complex of knots related to the above problem.

Definition 1.2. The *C_k-Gordian complex* $\mathcal{G}_{C_k}$ of knots is the simplicial complex defined by the following;
(1) the vertex set of $\mathcal{G}_{C_k}$ consists of all oriented knot types in S^3, and
(2) a family of $n + 1$ vertices $\{K_0, K_1, \ldots, K_n\}$ spans an n-simplex if and only if the C_k-distance $d_{C_k}(K_i, K_j) = 1$ for any i, j $\quad (i \neq j, i, j = 0, 1, \ldots, n)$.

The C_1-Gordian complex of knots corresponds to the Gordian complex in [7]. In this paper, we show Theorem 1.3.

Theorem 1.3. *Let k and n be integers more than 2 and 1, respectively. For any 0-simplex s of the C_k-Gordian complex, there exists an n-simplex σ such that s is a subcomplex of σ.*

Corollary 1.4. *Let k be an integer more than 2. For any knot K_0 and any given natural number n, there exists a family of knots $\{K_0, K_1, \ldots, K_n\}$ which satisfies the following:*
(1) *for any Vassiliev invariant v of order less than k, $v(K_i) = v(K_j)$, and*
(2) $d_{C_k}(K_i, K_j) = 1 \quad (i \neq j, i, j = 0, 1, \ldots, n)$.

Theorem 1.3 concerns the C_k-Gordian complex in the case that k is more than or equal to 3. In section 2, we describe the results for the Gordian complex and the C_2-Gordian complex, and in section 3, we give the proof of Theorem 1.3.

2. The Gordian complex and the C_2-Gordian complex of knots

The C_1-Gordian complex of knots corresponds to the Gordian complex in [7]. M. Hirasawa and Y. Uchida showed the following results.

Theorem 2.1([7]). *For any 1-simplex e of the Gordian complex, there exists an infinitely high dimensional simplex σ such that e is a subcomplex of σ.*

Corollary 2.2([7]). *For any knot K_0 and any given natural number n, there exists a family of knots $\{K_0, K_1, \ldots, K_n\}$ such that the Gordian distance $d_G(K_i, K_j) = 1$ for any i, j $(i \neq j, i, j = 0, 1, \ldots, n)$.*

In [7], M. Hirasawa and Y. Uchida construct the knots $K_1, \ldots, K_n$ and prove that they are distinct knots by calculating the coefficient polynomial of the skein polynomial([8]).

Different from the case of the Gordian complex, we have Proposition 2.3 for the C_2-Gordian complex.

Proposition 2.3. *There exists no 2-simplex in the C_2-Gordian complex.*

Proof. Suppose that there exists a 2-simplex $\{K_0, K_1, K_2\}$. By the result of M. Okada in [17], if the knot K_1 is obtained from the knot K_0 by a single delta move, then $a_2(K_1) = a_2(K_0) \pm 1$, where $a_2(K)$ is the second coefficient of the Conway polynomial of K. The C_2-move is the same move as the delta move. Since $d_{C_2}(K_0, K_1) = 1$, then

$$a_2(K_0) \neq a_2(K_1) \quad (mod\ 2).$$

Similarly,

$$a_2(K_0) \neq a_2(K_2) \quad (mod\ 2).$$

Therefore we have

$$a_2(K_1) = a_2(K_2) \quad (mod\ 2).$$

However, from $d_{C_2}(K_1, K_2) = 1$,

$$a_2(K_1) \neq a_2(K_2) \quad (mod\ 2).$$

This is a contradiction and completes the proof. □

3. Proof of Theorem 1.3.

In this section we consider the case that a 0-simplex is the trivial knot. If we show Theorem 1.3 in the case that a 0-simplex is the trivial knot, by considering the connected sum of it with a given knot K_0, we can obtain Theorem 1.3.

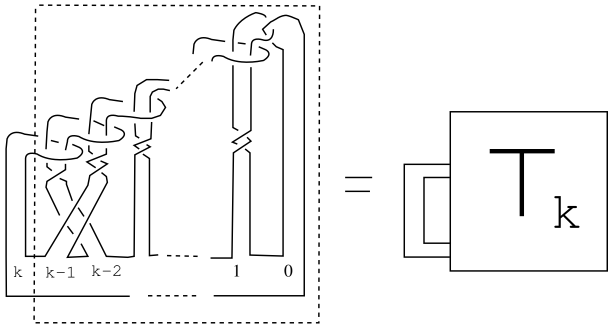

Fig. 3.1.

The knot in the left side of Fig. 3.1 is a trivial knot. We note that if we operate a C_k-move once on the knot in Fig. 3.1, it remains a trivial knot([15]). Let the knot in Fig. 3.2 be K_i, which is constructed by using i T_k's in Fig. 3.1. The knot K_0 is a trivial knot and the family $\{K_0, K_1, \ldots, K_n\}$ satisfies the condition $d_{C_k}(K_i, K_j) \leq 1$ for any i, j $(i \neq j, i, j = 0, 1, \ldots, n)$.

From now, we will show that K_i and K_j are different knot types by calculating the HOMFLY polynomial in the convention of W. B. R. Lickorish and K. Millett in [9].

The *HOMFLY polynomial* $P_K(\ell, m) \in Z[\ell^{\pm 1}, m^{\pm 1}]$([9]) is an invariant of an oriented link K, which is defined by the following:

(1) $\quad P_O(\ell, m) = 1$ and

(2) $\quad \ell P_{K_+}(\ell, m) + \ell^{-1} P_{K_-}(\ell, m) + m P_{K_0}(\ell, m) = 0,$

where O is the trivial knot and K_+, K_- and K_0 are diagrams of three links which

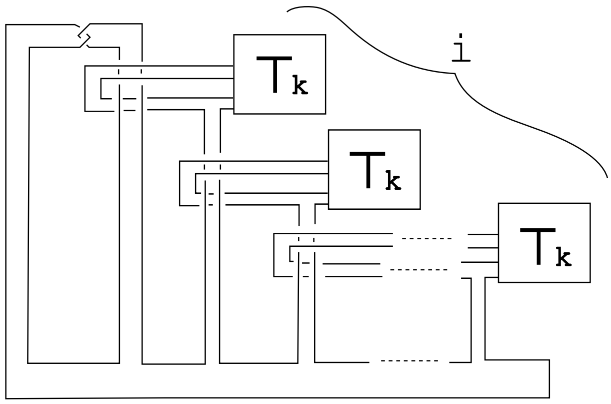

Fig.3.2.

are identical except near one point as shown in Fig. 3.3.

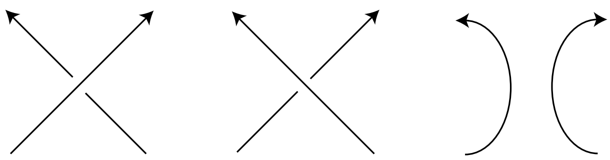

Fig.3.3.

Proposition 3.1([9]). *Let A and B be two string tangles. By $A+B$, A^N and A^D, we denote the tangle and the links in Fig. 3.4, respectively. Then*

$$(\mu^2-1)P_{(A+B)^N} = \mu(P_{A^N}P_{B^N} + P_{A^D}P_{B^D}) - (P_{A^N}P_{B^D} + P_{A^D}P_{B^N}),$$

where $\mu = -(\ell+\ell^{-1})m^{-1}$.

From now, we consider the one variable polynomial that is obtained from the HOMFLY polynomial by substituting $\ell = 1$ and it is also denoted by P_K. Let B be the tangle contained in the knot K_i as in Fig. 3.5 and let L_i be the link that is obtained from K_i by changing the tangle B to the tangle B' in Fig. 3.5. If we delete the tangle B from the knot K_i, we have the two string tangle and denote it by A. Then, $(A+B)^N = K_i$, $A^N = L_{i-1}$ and $A^D = K_{i-1}$. The knot B^N is a trivial knot and by $L'(k)$ we denote the link B^D. By Proposition 3.1, we have

$$\begin{aligned}(\mu^2-1)P_{K_i} &= \mu(P_{L_{i-1}} + P_{K_{i-1}}P_{L'(k)}) - (P_{L_{i-1}}P_{L'(k)} + P_{K_{i-1}}) \\ &= (\mu - P_{L'(k)})P_{L_{i-1}} + (\mu P_{L'(k)} - 1)P_{K_{i-1}}. \qquad (1)\end{aligned}$$

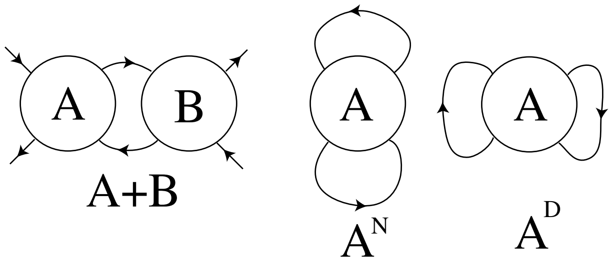

Fig.3.4.

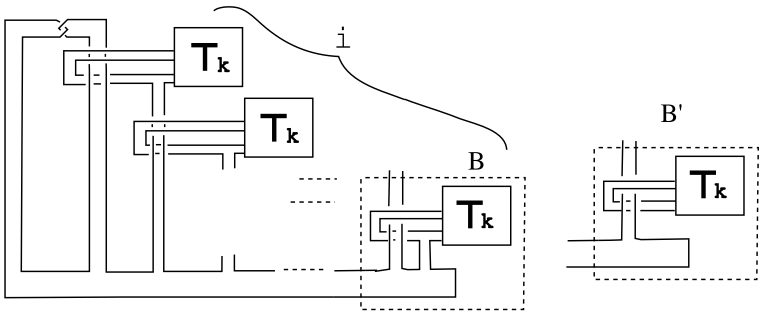

Fig.3.5.

The link B'^N is a trivial 2-component link. We denote the link B'^D by $\ell'(k)$, and for the link L_i we have

$$\begin{aligned}(\mu^2-1)P_{L_i} &= \mu(\mu P_{L_{i-1}}+P_{K_{i-1}}P_{\ell'(k)})-(P_{L_{i-1}}P_{\ell'(k)}+\mu P_{K_{i-1}}) \\ &= (\mu^2-P_{\ell'(k)})P_{L_{i-1}}+\mu(P_{\ell'(k)}-1)P_{K_{i-1}}. \qquad (2)\end{aligned}$$

From (1) and (2), it follows that

$$(\mu^2-1)P_{K_{i+1}}=(\mu P_{L'(k)}-1+\mu^2-P_{\ell'(k)})P_{K_i}-(\mu P_{L'(k)}-P_{\ell'(k)})P_{K_{i-1}},$$

that is,

$$(\mu^2-1)(P_{K_{i+1}}-P_{K_i})=(\mu P_{L'(k)}-P_{\ell'(k)})(P_{K_i}-P_{K_{i-1}}).$$

Therefore

$$P_{K_{i+1}}-P_{K_i}=(\frac{\mu P_{L'(k)}-P_{\ell'(k)}}{\mu^2-1})^{i-1}(P_{K_2}-P_{K_1}). \qquad (3)$$

From (1),

$$\begin{aligned}(\mu^2-1)(P_{K_2}-P_{K_1}) &= (\mu-P_{L'(k)})P_{L_1}+\{(\mu P_{L'(k)}-1)-\mu^2+1\}P_{K_1} \\ &= (\mu-P_{L'(k)})(P_{L_1}-\mu P_{K_1}).\end{aligned}$$

By the skein relation of the HOMFLY polynomial, we have $P_{L_1} = -\mu - mP_{\ell'(k)}$ and $P_{K_1} = -1 - mP_{L'(k)}$. Then,

$$P_{K_2} - P_{K_1} = (\frac{\mu - P_{L'(k)}}{\mu^2 - 1})m(\mu P_{L'(k)} - P_{\ell'(k)}).$$

By substituting the above formula for (3), it follows

$$P_{K_{i+1}} - P_{K_i} = (\frac{\mu P_{L'(k)} - P_{\ell'(k)}}{\mu^2 - 1})^i m(\mu - P_{L'(k)}).$$

Therefore we obtain

$$P_{K_i} = 1 + m(\mu - P_{L'(k)}) \sum_{t=0}^{i-1} (\frac{\mu P_{L'(k)} - P_{\ell'(k)}}{\mu^2 - 1})^t. \tag{4}$$

When $\ell = 1$, then $\mu = -\dfrac{2}{m}$. Calculating the polynomial by using the skein relation, we have the following formulas.

$$\begin{aligned}
P_{L'(k)} &= -\frac{2}{m} + (2 - m^2)^{k-3}(P_{L'(3)} + \frac{2}{m}). \\
P_{L'(3)} &= -\frac{2}{m} + (2 - m^2)^2 m(m + 2). \\
P_{\ell'(k)} &= \frac{4}{m^2} + (2 - m^2)^{k-3}(P_{\ell'(3)} - \frac{4}{m^2}). \\
P_{\ell'(3)} &= \frac{4}{m^2} + (2 - m^2)^2(-2m^2)(m + 2).
\end{aligned}$$

From $P_{L'(3)} + \dfrac{2}{m} \neq 0$, we have $\mu - P_{L'(k)} \neq 0$, and from the above formulas, the highest degree of m in $\mu P_{L'(k)} - P_{\ell'(k)}$ is $2k+1$. By (4) , we have that if $i \neq j$, then $P_{K_i} \neq P_{K_j}$ and this completes the proof. □

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