A two dimensional lattice of knots by C_{2n}-moves

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Abstract

We consider a local move on a knot diagram, where we denote the local move by M. If two knots K_1 and K_2 are transformed into each other by a finite sequence of M-moves, the M-distance between K_1 and K_2 is the minimum number of times of M-moves needed to transform K_1 into K_2. A M-distance satisfies the axioms of distance.

A two dimensional lattice of knots by M-moves is the two dimensional lattice graph which satisfies the following: The vertex set consists of oriented knots and for any two vertices K_1 and K_2, the distance on the graph from K_1 to K_2 coincides with the M-distance between K_1 and K_2, where the distance on the graph means the number of edges of the shortest path which connects the two knots.

Local moves called C_n-moves are closely related to Vassiliev invariants.

In this paper, we show that for any given knot K, there is a two dimensional lattice of knots by C_{2n}-moves with the vertex K.

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1 Introduction

Let M be a local move on a knot diagram. If two knots K_1 and K_2 are transformed into each other by a finite sequence of M-moves, $d_M(K_1, K_2)$ denotes the minimum number of times of M-moves needed to transform K_1 into K_2 and it is called *the M-distance between K_1 and K_2*.

We consider a lattice in R^2. *A two dimensional lattice graph* is an infinite graph whose vertices are lattice points in R^2 satisfying that two vertices are connected by an edge if and only if Euclidean distance between the pair is equal to one.

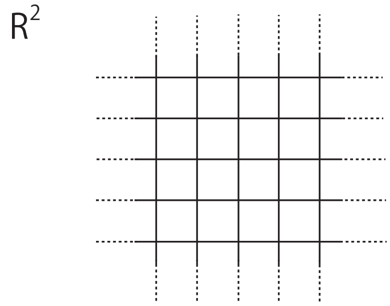

Fig. 1.

A two dimensional lattice of knots by M-moves is the two dimensional lattice graph which satisfies the following :
(1)The vertex set consists of oriented knots.
(2)For any two vertices K_1 and K_2, $d(K_1, K_2) = d_M(K_1, K_2)$, where $d(K_1, K_2)$ means the distance on the graph, that is, the number of edges of the shortest path which connects K_1

and K_2.

We consider a C_n-move as a local move. A C_n-move is closely related to Vassiliev invariants. When we have a knot invariant v which takes values in some abelian group, we may define an invariant of singular knots by the Vassiliev skein relation:

$$v(K_D) = v(K_+) - v(K_-),$$

where *a singular knot* is an immersion of a circle into R^3 whose singularities are transversal double points only and K_D, K_+ and K_- denote the diagrams of singular knots which are identical except near one point as is shown in Fig. 2. An invariant v is called *a Vassiliev invariant of order* n and denoted by v_n, if n is the smallest integer such that v vanishes on all singular knots with more than n double points([3]).

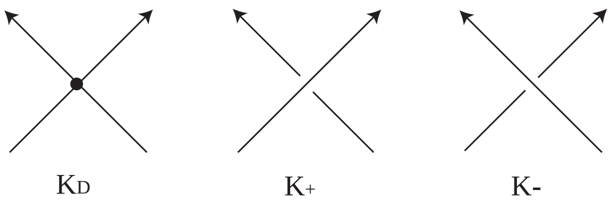

Fig. 2.

A standard C_n*-move* is a local move depicted in Fig. 3. *A* C_1*-move* is defined as a usual crossing change and a C_2-move is the same move as a Delta move([11, 12]). C_n-moves are defined as a family of local moves and any kind of C_n-move can be realized by a finite sequence of standard C_n-moves. For details, refer to [16] or [20]. Two knots are called C_n-equivalent if they can be transformed into each other by a finite sequence of standard C_n-moves. M. N. Goussarov([4]) and K. Habiro([6]) showed the following theorem independently.

Theorem 1.1[4, 6]. *Two knots are* C_n*-equivalent if and only if they have the same Vassiliev invariants of order less than* n*.*

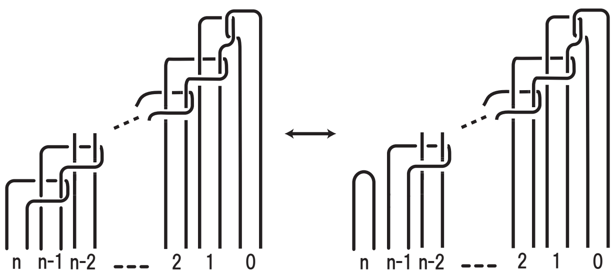

Fig. 3

In this paper, we show Theorem 1.2.

Theorem 1.2. *Let n be an integer more than one. For any given knot K, there exists a two dimensional lattice of knots by C_{2n}-moves with the vetex K.*

Theorem 1.2 is rewritten in terms of an isometric embedding between metric spaces.

Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is *an isometric embedding* if $d_Y(f_{(x_1)}, f_{(x_2)}) = d_X(x_1, x_2)$ for all x_1 and x_2 in X.

By L_2, we denote the set of all lattice points in R^2. For $\mathbf{a} = (x_1, y_1), \mathbf{b} = (x_2, y_2) \in L_2$, let $d : L_2 \times L_2 \to R$ be a map given by $d(\mathbf{a}, \mathbf{b}) = |x_1 - x_2| + |y_1 - y_2|$, then (L_2, d) is a metric space. Here, d coincides with the distance between two vertices on the two dimensional lattice graph.

Let Γ be a C_{2n}-equivalence class ($n > 1$), then we have a metric space $(\Gamma, d_{C_{2n}})$. By Theorem 1.2, we have Corollary 1.3.

Corollary 1.3. *There exists an isometric embedding from (L_2, d) to $(\Gamma, d_{C_{2n}})$.*

For any local move, we may define a graph whose vertices are oriented knots satisfying that two vertices are connected if and only if the distance by the local move between the two knots is equal to one.

Hirasawa and Uchida defined the Gordian complex of knots [7]. Generally, the λ-Gordian complex is defined for a local move λ [13]. *The λ-Gordian complex* is the simplicial complex defined by the following:
(1) the vertex set consists of oriented knots, and
(2) $n+1$ vertices $K_0, K_1, \ldots, K_n$ span an n-simplex if $d_\lambda(K_i, K_j) = 1$ for $i \neq j, i, j = 0, 1, \ldots, n$.

The 1-skelton of the λ-Gordian complex is called *the λ-Gordian graph.* Ichihara and Jong[9] defined the (ι, λ)-Gordian complex for a knot invariant ι and a local move λ. The vertex set of it consists of the equivalence classes of knots, where two knots are equivalent if they have the same ι. In the case that ι is a Conway polynomial and λ is a Delta move, they show that the (ι, λ)-Gordian graph is Gromov hyperbolic as a geodesic space. For detail, see [9]. Theorem 1.2 shows that the C_{2n}-Gordian graph is not Gromov hyperbolic.

2 C_n-moves and knot invariants

By K^m, we denote a singular knot with m double points. From the definition of the Vassiliev invariant, $v_m(K^m)$ does not change by a crossing change and it is determined by the positions of the double points on K^m. To show the positions of double points, the notion of a chord diagram is introduced in [3]. *A chord diagram of order n* is an oriented circle with n chords. By connecting the preimages of each double point by a chord, we may associate the chord diagram to a singular knot. The value of v_n for a chord diagram of order n is defined as the value of it for a singular knot with n double points that is associated with the chord diagram. In the additive group generated by the chord diagrams of order n, the relation in Fig. 3 is called *the $4T$ relation.*

Chord diagrams are generalized to Jacobi diagrams in [1]. *A Jacobi diagram of order n* is a trivalent graph with $2n$ vertices. It is a union of a circle and an internal graph G. The circle is oriented and the other edges are all unoriented. Each trivalent vertex on G has an orientation, that is a cyclic ordering of the edges incident to it. In the additive group generated by the Jacobi diagrams of order n, the relation in Fig. 4 is called *the STU relation.*

Let A_n be the additive group generated by the chord diagrams of order n modulo the $4T$ relation and B_n the additive group generated by the Jacobi diagrams of order n modulo the

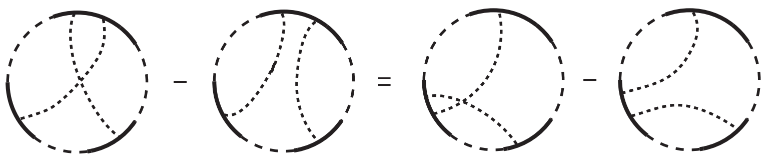

Fig. 3

STU relation. Then the isomorphism between A_n and B_n is induced by the inclusion of chord diagrams into Jacobi diagrams [1].

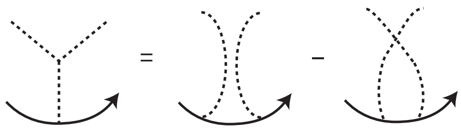

Fig. 4

A one-branch tree diagram T is a special kind of Jacobi diagram whose internal graph G is isomorphic to a standard n-tree in Fig. 5 preserving the vertex orientations([14]). Label the branches of the standard n-tree as in Fig. 5. We may label the branches of G under the isomorophism between G and the standard n-tree. And number the vertices of the circle of T by $0, 1, 2, \ldots, n$ in the counterclockwise direction such that the vertex on the circle which corresponds to the branch 0 of G is numbered by 0. The correspondence between the labels of branches of G and the numbers of the corresponding vertices on the circle determines a permutation $\sigma \in S_n$. Conversely, if a permutation $\sigma \in S_n$ is given, a unique one-branch tree diagram T can be constructed. Then we denote a one-branch tree diagram by T_σ. By STU relations, a one-branch tree diagram can be expressed as a linear combination of chord diagrams. The value of v_n for a one-branch tree diagram of order n is defined as the linear combination of the values for the chord diagrams.

Fig. 5

A one-branch tree diagram is closely related to a standard C_n-move.

Theorem 2.1[17]. *If a knot K' is obtained from a knot K by a single standard C_n-move, then*

$$v_n(K') - v_n(K) = \pm v_n(T_\sigma),$$

where T_σ is a one-branch tree diagram of order n.

In Theorem 2.1, the one-branch tree diagram T_σ is determined by the positions of arcs on a knot K in the performed C_n-move and the sign of the formula depends only on the orientations of arcs in the C_n-move.

From here, we describe the results for concrete Vassiliev invariants.

The HOMFLY polynomial $P(L; t, z) \in \mathbb{Z}[t^{\pm 1}, z^{\pm 1}]$ is an invariant of the isotopy type of an oriented knot or link L, which is defined by the following formulas:
(i) $P(U; t, z) = 1$;
(ii) $t^{-1}P(L_+; t, z) - tP(L_-; t, z) = zP(L_0; t, z)$,
where U is a trivial knot and L_+, L_- and L_0 are three links that are identical except near one point where they are as shown in Fig. 6.

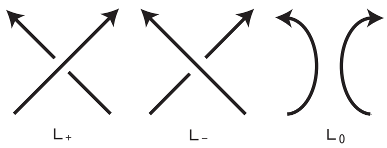

Fig. 6.

If we arrange the HOMFLY polynomial by the variable z, then the coefficient of z^k is the polynomial of t, which is called the P_k polynomial. Let $P_k^{(n)}(L; 1)$ be the nth derivative of the P_k polynomial of the HOMFLY polynomial of a link L evaluated at 1. Kanenobu and Miyazawa showed Theorem 2.2.

Theorem 2.2 [10] $P_k^{(n)}(L; 1)$ *is a Vassiliev invariant of order max* $\{n + k, 0\}$.

For $P_0^{(n)}(K, 1)$ of a knot K, the first author obtained the following theorem.

Theorem 2.3 [8] *Let n be an integer more than or equal to 3. Let K and K' be oriented knots such that K' is obtained from K by a single standard C_n-move. Then,*

$$P_0^{(n)}(K; 1) - P_0^{(n)}(K'; 1) = \begin{cases} 0 & \text{if } T_\sigma \text{ is a non-planar graph,} \\ \pm n! \cdot 2^n & \text{if } T_\sigma \text{ is a planar graph.} \end{cases}$$

Here T_σ is the one-branch tree diagram of order n which corresponds to a performed C_n-move.

By substituting $t = 1$ for the HOMFLY polynomial, we have the Conway polynomial $\Delta(z)$. It is well known that the coefficient of z^n of the Conway polynomial is a Vassiliev invariant of order n. The following result is shown in the proof of the main theorem in [18]

Lemma 2.4 [18] *Let m' be an even natural number more than or equal to 4 and $a_{m'}$ the coefficient of $z^{m'}$ of the Conway polynomial. By T_σ, we denote a one-branch tree diagram of order m', where σ is assumed that $\sigma(m'-1) < \sigma(m')$ by the AS relation.*
Then $a_{m'}(T_\sigma) = \pm 2$ if and only if σ satisfies that $\sigma(m'-1) < \sigma(1) < \sigma(m')$ as shown in Fig. 7.

By Lemma 2.4, if T_σ is planar, $a_{m'}(T_\sigma) = 0$. From Theorem 2.1, Lemma 2.4 and Theorem 2.3, a C_{2n}-move cannot change both of the values of $P_0^{(2n)}(K, 1)$ and a_{2n}. Then we have

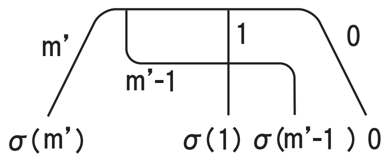

Fig. 7.

Corollary 2.5.

Corollary 2.5 *If a knot K_2 is obtained from a knot K_1 by a finite sequence of standard C_{2n}-moves, then*

$$d_{C_{2n}}(K_1, K_2) \geqq \frac{\mid P_0^{(2n)}(K_1;1) - P_0^{(2n)}(K_2;1) \mid}{(2n)! \cdot 2^{2n}} + \frac{\mid a_{2n}(K_1) - a_{2n}(K_2) \mid}{2}.$$

3 Proof of Theorem 1.2

We consider a 2 dimensional lattice graph and coordinates of K are assumed to be $(0,0)$. We perform two types of standard C_{2n}-moves to K as shown in Fig. 8. One is a C_n-move corresponding to a planar one-branch tree diagram and another is a C_n-move corresponding to the one-branch tree diagram as shown in Fig 7. x and y denote the number of times of full twists. If x and y are negative, they mean the inverse full twists as shown in Fig, 9.

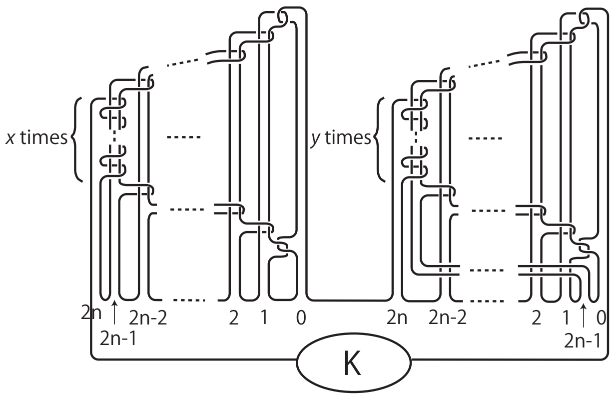

Fig. 8.

The knot as shown in Fig. 8 is made to correspond to the point whose coordinates are (x,y). Let K_1 be the knot at (x_1,y_1) and K_2 the knot at (x_2,y_2). By Theorem 2.3,

$$\mid x_1 - x_2 \mid = \frac{\mid P_0^{(2n)}(K_1;1) - P_0^{(2n)}(K_2;1) \mid}{(2n)! \cdot 2^{2n}}$$

.

By Theorem 2.1 and Lemma 2.4,

$$\mid y_1 - y_2 \mid = \frac{\mid a_{2n}(K_1) - a_{2n}(K_2) \mid}{2}.$$

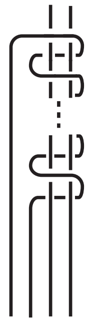

Fig. 9.

By Corollary 2.5,

$$d_{C_{2n}}(K_1, K_2) \geqq \mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid .$$

Performing standard C_{2n}-moves by $\mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid$ times, K_1 is transformed into K_2. Therefore,

$$d_{C_{2n}}(K_1, K_2) = \mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid .$$

This completes the proof of Theorem 1.2.

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