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A NUMERICAL INVARIANT FOR TWO COMPONENT SPATIAL GRAPHS

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Dedicated to Professor Akio Kawauchi for his 60th birthday.

ABSTRACT

We define an invariant for two component spatial graphs. Although the definition of the invariant is alike a linking number, it is different from the absolute value of a linking number. We show that the invariant is not a finite type invariant.

Keywords: Spatial graph; Linking number; Vassiliev type invariant.

Mathematics Subject Classification 2000: 57M25, 05C10

1. Introduction

Graphs are finite, consisting of a finite number of vertices and edges and considered as a one dimensional CW complex. By *a spatial embedding* we mean an embedding of a graph in R^3 and also an image of an embedding. An image of a spatial embedding is also called a spatial graph.

Let $G = G_1 \cup G_2$, where G_k $(k = 1, 2)$ is a connected graph such that the degrees of all vertices are in multiples of 4 and it has an Eulerian circuit which satisfies the following special condition. *An Eulerian circuit* is a circuit containing all edges: For any vertex v, starting from v, we walk along the Eulerian circuit and reach v for the first time. Then the length of the part of the Eulerian circuit which we walk is odd, where the length of a circuit is the number of edges of it.

Fig. 1 shows examples of G_k. We consider the spatial graphs obtained from polygons with odd vertices by doubling edges. They are examples of G_k. For Eulerian bipartite graphs, attach loops to their all vertices. Each number of the attached loops is half of the degree of the corresponding vertex. These are also examples of G_k.

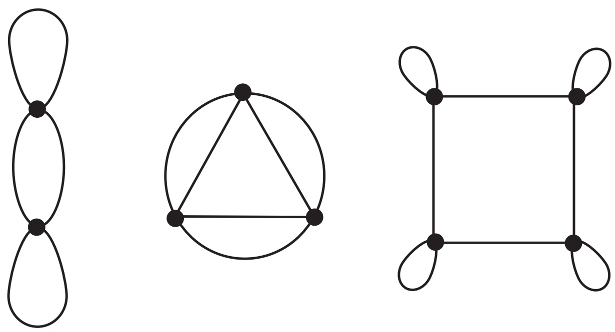

Fig. 1.

Let $f : G \to R^3$ be an embedding and $f(G) = \tilde{G} = \tilde{G}_1 \cup \tilde{G}_2$. We give an orientation to $\tilde{G}$ along the Eulerian circuit satisfying the above condition. By $M(\tilde{G}_1, \tilde{G}_2) = \{c_0, c_1, \ldots, c_m\}$, we denote the set of all crossing points between $\tilde{G}_1$ and $\tilde{G}_2$. When we walk from a crossing c_0 to a crossing c_i along the orientation on $\tilde{G}_k$, $d_{\tilde{G}_k}(c_0, c_i)$ means the number of vertices which we passed $(k = 1, 2)$. Let $d_{\tilde{G}}(c_0, c_i) = d_{\tilde{G}_1}(c_0, c_i) + d_{\tilde{G}_2}(c_0, c_i)$ and we define the number $H(G)$ by the following.

$$H(\tilde{G}) = \frac{1}{2}\left|\sum_{i=0}^{m}(-1)^{d_{\tilde{G}}(c_0,c_i)} sgn(c_i)\right|,$$

where $sgn(c_i)$ is the sign of a crossing c_i in Fig 2.

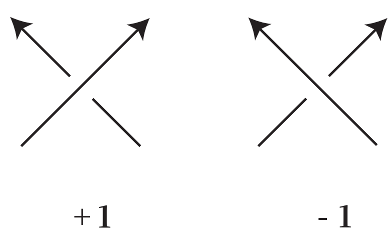

Fig. 2.

By *a diagram* of a spatial graph, we mean a regular projection on R^2 with over and under crossing information as a knot diagram. It is known that two diagrams represent ambient isotopic spatial graphs if and only if they are transformed into each other by a finite sequence of (extended) Reidemeister moves (I)∼(V) as is shown in Fig. 3 [1] [5].

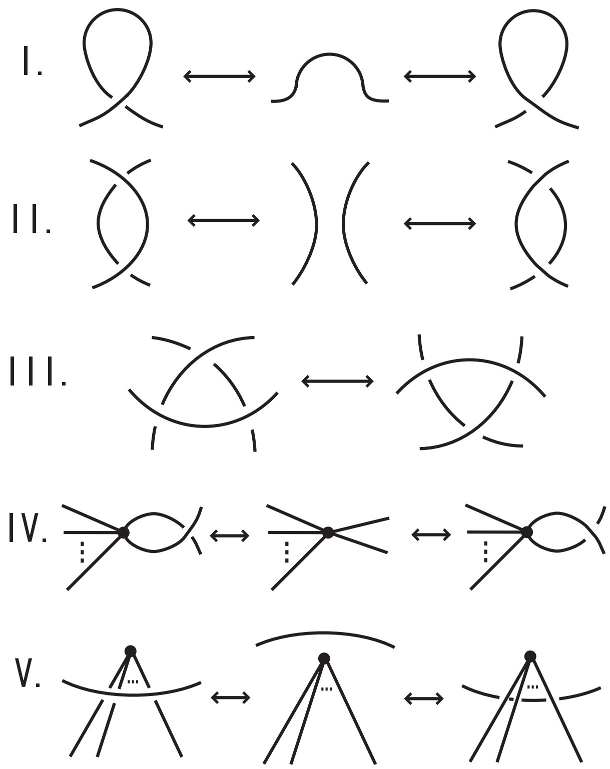

Fig. 3.

The number $H(\tilde{G})$ is independent of the choice of a base point c_0 and we show the following theorem.

Theorem 3.2. *The number* $H(\tilde{G})$ *is an ambient isotopy invariant for spatial graphs with special Eulerian circuits.*

As for knots, Vassiliev type invariants for spatial graphs are defined in [3] and [4]. In this paper, we follow the definition in [3].

Let G be a finite graph (with a special Eulerian circuit) and $SE(G)$ the set of all embeddings of G into R^3. Let $\mathcal{R}$ be a commutative ring with unit 1 and $v : SE(G) \to \mathcal{R}$ an ambient isotopy invariant. And let $SE_i(G)$ be the set of all i-singular embeddings of G into R^3, where an i-singular embedding is a continuous map whose multiple points are exactly i double points of edges spanning small flat planes. Such a double point is called *a crossing vertex*. Under a given edge orientation of G, v is extended to an ambient isotopy invariant $v_i : SE_i(G) \to \mathcal{R}$ by the following formula:

$$v_i(f_0) = v_{i-1}(f_+) - v_{i-1}(f_-)$$

where f_0, f_+ and f_- are related as is shown in Fig. 4. In Fig. 4, a gray vertex means a crossing vertex (a double point).

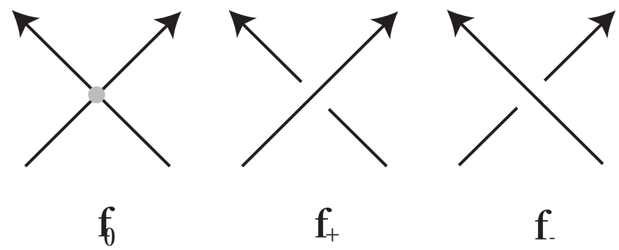

Fig. 4.

We only consider ambient isotopies that preserve a small flat plane at each crossing vertex. We say that v is *a Vassiliev type invariant of order* n if n is the smallest integer which satisfies that $v_{n+1} : SE_{n+1}(G) \to \mathcal{R}$ is a zero map. We note that the definition of a Vassiliev type invariant of order n is independent of the choice of edge orientations. A Vassiliev type invariant of order m for some integer m is called *a finite type invariant*. For the number $H(\tilde{G})$, we have Theorem 3.3.

Theorem 3.3 *The number $H(\tilde{G})$ is not a finite type invariant for spatial graphs with special Eulerian circuits.*

2. Linking numbers

We may define a linking number for two component spatial graphs. Let $G = G_1 \cup G_2$, where G_k $(k = 1, 2)$ is an Eulerian graph. We fix an Eulerian circuit and give an orientation on all edges of G_k along the Eulerian circuit. Let $f : G \to R^3$ be an embedding and $f(G) = \tilde{G} = \tilde{G}_1 \cup \tilde{G}_2$. By $M(\tilde{G}_1, \tilde{G}_2) = \{c_0, c_1, \ldots, c_m\}$, we denote the set of all crossing points between $\tilde{G}_1$ and $\tilde{G}_2$ as the previous section. *A linking number* $lk(\tilde{G}_1, \tilde{G}_2)$ between $\tilde{G}_1$ and $\tilde{G}_2$ is defined by the following [2].

$$lk(\tilde{G}) = lk(\tilde{G}_1, \tilde{G}_2) = \frac{1}{2}\sum_{i=0}^{m} sgn(c_i).$$

It is known that a linking number is a well-defined ambient isotopy invariant and we have the following.

Proposition 2.1 *A linking number $lk(\tilde{G}_1, \tilde{G}_2)$ is a Vassiliev type invariant of order 1.*

Proof. Let $\tilde{G}(d_1, d_2)$ be a 2-singular spatial embedding of G with crossing vertices d_1 and d_2. A spatial graph $\tilde{G}(\varepsilon_1, \varepsilon_2)$ is obtained from $\tilde{G}(d_1, d_2)$ by making a crossing vertex d_i into a crossing point signed ε_i $(i = 1, 2)$. By the definition of the Vassiliev type invariant,

$$v(\tilde{G}(d_1, d_2)) = v(\tilde{G}(+1, +1)) - v(\tilde{G}(+1, -1)) - v(\tilde{G}(-1, +1)) + v(\tilde{G}(-1, -1)) \quad (2.1)$$

Here we substitute $lk(\tilde{G})$ for v in (2.1).

In the case that both d_1 and d_2 are crossing vertices between $\tilde{G}_1$ and $\tilde{G}_2$, we have

$$lk(\tilde{G}(+1, +1)) = lk(\tilde{G}(-1, -1)) + 2$$

and

$$lk(\tilde{G}(+1, -1)) = lk(\tilde{G}(-1, +1)) = lk(\tilde{G}(-1, -1)) + 1.$$

By the above formulas, we have $v(\tilde{G}(d_1, d_2)) = 0$.

In the case that at least one of d_1 and d_2, for example d_1, is on the same component, we have

$$lk(\tilde{G}(+1, \varepsilon_2)) = lk(\tilde{G}(-1, \varepsilon_2)), (\varepsilon_2 = \pm 1).$$

Then we have $v(\tilde{G}(d_1, d_2)) = 0$. □

Example 2.2.
(i) Let $\tilde{G} = \tilde{G}_1 \cup \tilde{G}_2$ be the spatial graph as is shown in Fig. 5. When we walk along the Eulerian circuit, we trace edges in order of a, b and c. In this example, we have $lk(\tilde{G}) = 0$ and $H(\tilde{G}) = 2\ell$.

(ii) Let $\tilde{G}'$ be the spatial graph obtained from $\tilde{G}$ in Fig. 5 by changing crossing points such that the signs of all crossing points are $+1$. Then we have $lk(\tilde{G}') = 2\ell$ and $H(\tilde{G}') = 0$.

Remark 2.3.
If $G = G_1 \cup G_2$ is homeomorphic to $S^1 \cup S^1$ (and have no vertex), it follows that

$$d_{\tilde{G}_1}(c_0, c_i) + d_{\tilde{G}_2}(c_0, c_i) = 0.$$

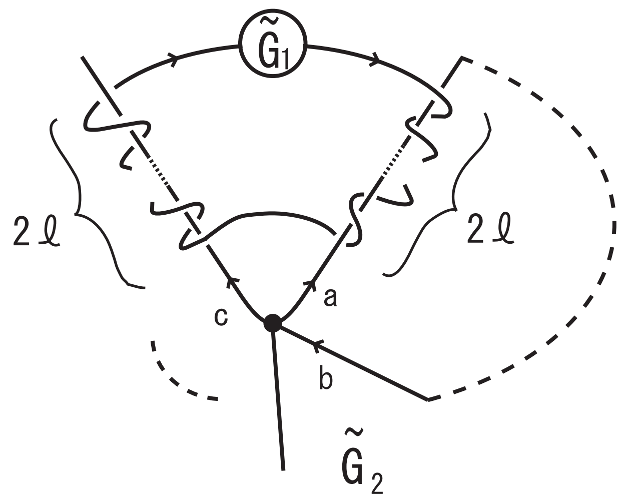

Fig. 5.

Then we have

$$H(\tilde{G}) = \frac{1}{2}\left|\sum_{i=0}^{m}(-1)^{d_{\tilde{G}}(c_0,c_i)} sgn(c_i)\right|$$
$$= \left|\frac{1}{2}\sum_{i=0}^{m} sgn(c_i)\right| = \left|lk(\tilde{G}_1, \tilde{G}_2)\right|.$$

3. Results and proofs

At first we show that $H(\tilde{G})$ is well-defined.

Proposition 3.1. *The number $H(\tilde{G})$ is independent of the choice of a base point c_0.*

Proof. Since the degree of each vertex in G_k $(k = 1, 2)$ is in multiples of 4, the number of edges in it is even. Starting at c_0, we walk the Eulerian circuit and reach c_0 on $\tilde{G}_k$. Then the number of vertices we pass is even. It follows that

$$d_{\tilde{G}_k}(c_0, c_k) + d_{\tilde{G}_k}(c_k, c_i) + d_{\tilde{G}_k}(c_i, c_0) \equiv 0 \quad (mod\ 2),$$

for $k = 1, 2$. Therefore

$$d_{\tilde{G}}(c_0, c_i) \equiv d_{\tilde{G}}(c_0, c_k) + d_{\tilde{G}}(c_k, c_i) \quad (mod\ 2).$$

We have

$$\begin{aligned} H(\tilde{G}) &= \frac{1}{2}\left|\sum_{i=0}^{m}(-1)^{d_{\tilde{G}}(c_0,c_i)}sgn(c_i)\right| \\ &= \frac{1}{2}\left|\sum_{i=0}^{m}(-1)^{d_{\tilde{G}}(c_0,c_k)+d_{\tilde{G}}(c_k,c_i)}sgn(c_i)\right| \\ &= \frac{1}{2}\left|(-1)^{d_{\tilde{G}}(c_0,c_k)}\sum_{i=0}^{m}(-1)^{d_{\tilde{G}}(c_k,c_i)}sgn(c_i)\right| \\ &= \frac{1}{2}\left|\sum_{i=0}^{m}(-1)^{d_{\tilde{G}}(c_k,c_i)}sgn(c_i)\right| \end{aligned}$$

□

Theorem 3.2. *The number $H(\tilde{G})$ is an ambient isotopy invariant for spatial graphs with special Eulerian circuits.*

Proof. It is clear that $H(\tilde{G})$ is invariant under Reidemeister moves I and IV.

For Reidemeister move II, if two arcs are in the same component, $H(\tilde{G})$ cannot be changed after operating the move by its definition. In the case that two arcs are contained in the different components, we have that

$$d_{\tilde{G}}(c_0,c_i) = d_{\tilde{G}}(c_0,c_j)$$

and

$$sgn(c_i) = -sgn(c_j),$$

where c_i and c_j are crossing points as is shown in Fig. 6 and c_0 is a base point. Therefore,

$$(-1)^{d_{\tilde{G}}(c_0,c_i)}sgn(c_i) + (-1)^{d_{\tilde{G}}(c_0,c_j)}sgn(c_j) = 0.$$

It follows that $H(\tilde{G})$ cannot be changed by Reidemeister move II.

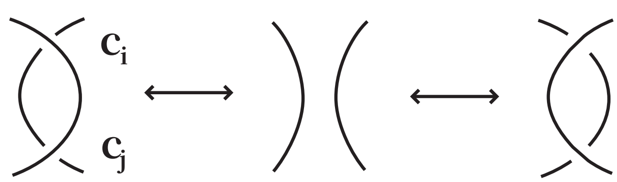

Fig. 6.

For Reidemeister move III, it is enough to consider two cases; the case that three arcs are on the same component and the other case that two arcs are on one component and the third is on the other. If three arcs are on the same component,

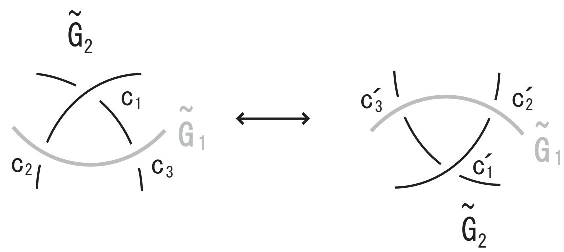

Fig. 7.

$H(\tilde{G})$ cannot be changed by the definition. In the case that two arcs are on the same component and the third is on the other, for example, the case in Fig. 7,

$$d_{\tilde{G}}(c_0, c_k) = d_{\tilde{G}}(c_0, c'_k)$$

and

$$sgn(c_k) = sgn(c'_k) \quad (k = 2, 3).$$

Therefore

$$\begin{aligned} &(-1)^{d_{\tilde{G}}(c_0,c_2)} sgn(c_2) + (-1)^{d_{\tilde{G}}(c_0,c_3)} sgn(c_3) \\ = &(-1)^{d_{\tilde{G}}(c_0,c'_2)} sgn(c'_2) + (-1)^{d_{\tilde{G}}(c_0,c'_3)} sgn(c'_3). \end{aligned}$$

It follows that $H(\tilde{G})$ cannot be changed in Fig. 7. For the other cases, we can show that $H(\tilde{G})$ cannot be changed similarly.

From here, we consider Reidemeister move V. Since $H(\tilde{G})$ is invariant under Reidemeister move IV, we may assume the following; when we walk along the Eulerian circuit starting at the vertex v and reach v for the first time, we trace adjacent edges e_i and e_j as is shown in Fig. 8.

From the property of the Eulerian circuit, the length of the part of the Eulerian circuit that we walk from v to v in Fig. 8 is odd. Then the number of vertices between c_i and c_j on the part of Eulerian circuit is even. It follows that

$$d_{\tilde{G}_1}(c_0, c_i) \equiv d_{\tilde{G}_1}(c_0, c_j) \quad (mod\ 2).$$

And

$$d_{\tilde{G}_2}(c_0, c_i) \equiv d_{\tilde{G}_2}(c_0, c_j) \quad (mod\ 2).$$

Then we have

$$d_{\tilde{G}}(c_0, c_i) \equiv d_{\tilde{G}}(c_0, c_j) \quad (mod\ 2).$$

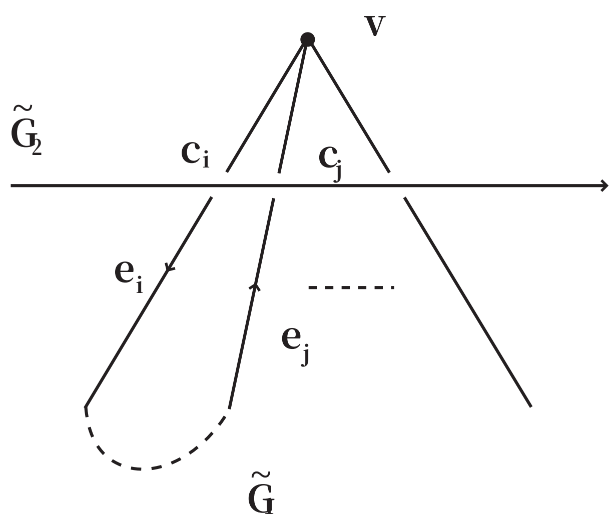

Fig. 8.

For signs of crossings, we have

$$sgn(c_i) = -sgn(c_j).$$

Therefore, we have

$$(-1)^{d_{\tilde{G}}(c_0,c_i)} sgn(c_i) + (-1)^{d_{\tilde{G}}(c_0,c_j)} sgn(c_j) = 0.$$

By considering for another pair of adjacent crossings similarly, $H(\tilde{G})$ cannot be changed by Reidemeister move V. This completes the proof of Theorem 3.2. □

Theorem 3.3. *The number $H(\tilde{G})$ is not a finite type invariant for spatial graphs with special Eulerian circuits.*

Proof. Let $\tilde{G} = \tilde{G}_1 \cup \tilde{G}_2$ be the singular spatial graph as is shown in Fig. 9. When we walk along the Eulerian circuit, we trace edges in order of a, b and c. There are a crossing and a crossing vertex (a double point) between $\tilde{G}_1$ and $\tilde{G}_2$ on the edge labeled a, and there are $n-1$ crossings and $n-1$ crossing vertices on c.

By $\tilde{G}(1, 2, \cdots, n)$, we denote $\tilde{G}$ in Fig. 9, where the numbers mean crossing vertices. Let $\tilde{G}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ be the spatial graph obained from $\tilde{G}(1, 2, \cdots, n)$ by making a crossing vertex i into a crossing point signed ε_i ($\varepsilon_i = \pm 1, i = 1, 2, \cdots, n$).

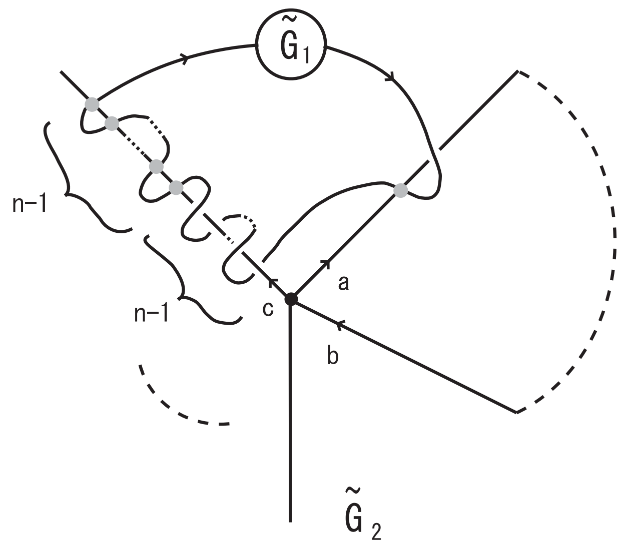

Fig. 9.

By the definition of the Vassiliev type invariant,

$$\begin{aligned} H(\tilde{G}) &= \sum_{\varepsilon_i=\pm 1, i=1,2,\cdots,n} (-1)^{n(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)} H(\tilde{G}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)) \\ &= \sum_{\varepsilon_i=\pm 1, i=2,\cdots,n} \{(-1)^{n(\varepsilon_2,\cdots,\varepsilon_n)} H(\tilde{G}(+1,\varepsilon_2,\cdots,\varepsilon_n)) \\ &\quad + (-1)^{1+n(\varepsilon_2,\cdots,\varepsilon_n)} H(\tilde{G}(-1,\varepsilon_2,\cdots,\varepsilon_n))\}, \end{aligned}$$

where $n(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)$ means the number of -1 in $\{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n\}$. From here, let $j = n(\varepsilon_2,\cdots,\varepsilon_n)$ and $\tilde{G}(\varepsilon_1; j) = \tilde{G}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)$. The above formula is rewritten to the following.

$$H(\tilde{G}) = \sum_{j=0}^{n-1}(-1)^j \ {}_{n-1}C_j H(\tilde{G}(+1;j)) + \sum_{j=0}^{n-1}(-1)^{j+1} \ {}_{n-1}C_j H(\tilde{G}(-1;j)) \quad (3.2)$$

Since the number of vertices from a to b along the Eulerian circuit is even by the property of the Eulerian circuit, we have

$$H(\tilde{G}(+;j)) = \frac{1}{2}\left|2-(n-1+n-1-2j)\right| = \begin{cases} n-2-j & (j=0,1,\cdots,n-2) \\ 1 & (j=n-1) \end{cases}$$

and

$$H(\tilde{G}(-;j)) = \frac{1}{2}\left|n-1+n-1-2j\right| = n-1-j \quad (j=0,1,\cdots,n-1).$$

By (3.2) and the above formulas, we have

$$
\begin{aligned}
H(\tilde{G}) = {}& {}_{n-1}C_{n-1}(-1)^{n-1} \cdot 1 + \sum_{j=0}^{n-2} {}_{n-1}C_j(-1)^j(n-2-j) \\
&+ \sum_{j=0}^{n-2} {}_{n-1}C_j(-1)^{j+1}(n-1-j) \\
&= (-1)^{n-1} + \sum_{j=0}^{n-2} {}_{n-1}C_j(-1)^j(n-1-j-1) - \sum_{j=0}^{n-2} {}_{n-1}C_j(-1)^j(n-1-j) \\
&= (-1)^{n-1} - \sum_{j=0}^{n-2} {}_{n-1}C_j(-1)^j \\
&= (-1)^{n-1} - (\sum_{j=0}^{n-1} {}_{n-1}C_j(-1)^j - {}_{n-1}C_{n-1}(-1)^{n-1})
\end{aligned}
$$

By $\sum_{j=0}^{n-1} {}_{n-1}C_j(-1)^j = \{1+(-1)\}^{n-1} = 0$, we have

$$H(\tilde{G}) = 2(-1)^{n-1} \neq 0.$$

This completes the proof of Theorem 3.3. □

References

[1] L. H. Kauffman, Invariants of graphs in three-space, *Trans. Amer. Math. Soc.*, **311(2)** (1989), 697-710.
[2] Y. Ohyama, Local moves on a graph in R^3, *J. Knot Theory Ramifications*, **5(2)** (1996), 265-277.
[3] Y. Ohyama and K. Taniyama, Vassiliev invariant of knots in a spatial graph, *Pacific J. Math.*, **200(1)** (2001), 191-205.
[4] T. Stanford, Finite type invariants of knots, links and graphs, *Topology*, **35(4)** (1996), 1027-1050.
[5] S. Yamada, An invariant of spatial graphs, *J. Graph Theory* **13(4)** (1989), 537-551.