Solitons in Supersymmetric Gauge Theories: 
(Walls, Vortices, Monopoles, Instantons and Their Composites)

Norisuke Sakai (Tokyo Institute of Technology) In collaboration with
M. Arai, M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta,
Y. Tachikawa, D. Tong, and Y. Yang,

Talk at Sapporo workshop 2005.10.26-28

Contents

1 Discrete Vacua in Higgs Phase and BPS Eq. 2
2 BPS Wall Solutions 5
3 Global Structure of Wall Moduli Space 8
4 Webs of Domain Walls 11
5 Composite Soliton of Wall, Vortex, and Monopole 16
6 Conclusion 20
1 Discrete Vacua in Higgs Phase and BPS Eq.

Brane-World = Our world on a Topological Defect in higher dim. spacetime

Topological Defects: Walls, Vortices, ... are preferably Solitons

Part of Supersymmetry (SUSY) preserved → BPS state

Solves the Equation of Motion

Soliton Dynamics: Important for Nonperturbative Effects

Supersymmetry (SUSY) helps

to obtain realistic unified models with $\mathcal{N} = 1$ SUSY

to find Solitons (Walls, Vortices, ...) as BPS states

Parameters of the Solution = Moduli

→ Massless fields on the world volume
SUSY $U(N_C)$ Gauge Theory with $N_F$ Flavors

5 Dimensions $\rightarrow$ 8 SUSY $(M, N, \cdots = 0, 1, 2, 3, 4)$

Vector multiplets: $W_M$ Gauge field, $\Sigma$ Real Scalar ($N_C \times N_C$ matrix)

Hypermultiplets: $(H^i)^{rA} \equiv H^{irA}$ Complex Scalar ($N_C \times N_F$ matrix)

$(i = 1, 2$; Color $r = 1, \cdots, N_C$; Flavor $A = 1, \cdots, N_F$)

Minimal kinetic terms

- Gauge coupling $g$ for $U(N_C)$, Fayet-Iliopoulos (FI) parameter $c$
- Hypermultiplet Mass $(M)^A_B \equiv m_A \delta^A_B$

\[
\mathcal{L} = -\frac{1}{2g^2} \text{Tr}(F_{MN}(W)F^{MN}(W)) + \frac{1}{g^2} \text{Tr}(D^M \Sigma D_M \Sigma) + \text{Tr} \left[ D^M H^i (D_M H^i)^\dagger \right] - V
\]

\[
V = \frac{g^2}{4} \text{Tr} \left[ (H^1 H^{1\dagger} - H^2 H^{2\dagger} - c1_{N_C})^2 + 4H^2 H^{1\dagger} H^1 H^{2\dagger} \right] + \text{Tr} \left[ (\Sigma H^i - H^i M)(\Sigma H^i - H^i M)^\dagger \right]
\]

Non-degenerate mass: $m_A > m_{A+1} \rightarrow$ Flavor symmetry: $U(1)_F^{N_F-1}$
Discrete SUSY Vacua: Color-flavor locking $\langle A_1 A_2 \cdots A_{N_C} \rangle$

$H^{1rA} = \sqrt{c} \delta^{A_r} A, \quad H^{2rA} = 0, \quad \Sigma = \text{diag}(m_{A_1}, \cdots, m_{A_{N_C}})$

Number: $\frac{N_F!}{(N_F-N_C)!N_C!} \sim e^{N_F \log(x^{-x}(1-x)^{-(1-x)})}, \quad x \equiv N_C/N_F$

1/2 BPS Equations

Dependence on $y \equiv x^4$, 4 D Poincaré invariance $\rightarrow W_{M\neq y} = 0$

1/2 BPS equations $\iff$ Conserved SUSY: $\gamma^4 \varepsilon^i = -i(\sigma^3)^{ij} \varepsilon^j$

$\mathcal{D}_y \Sigma = g^2 \left( c1_{N_C} - H^1 H^1 + H^2 H^2 \right)/2, \quad 0 = g^2 H^1 H^2$

$\mathcal{D}_y H^1 = -\Sigma H^1 + H^1 M, \quad \mathcal{D}_y H^2 = \Sigma H^2 - H^2 M$
Topological sector labeled by \( \langle A_1 A_2 \cdots A_{N_C} \rangle \leftarrow \langle B_1 B_2 \cdots B_{N_C} \rangle \)

2 BPS Wall Solutions

Solving BPS Equations

\( S(y) \in GL(N_C, C) \) defined by: \( \Sigma + iW_y \equiv S^{-1}(y)\partial_y S(y) \)

BPS Eqs. for Hypermultiplet can be solved by

\[
H^1(y) = S^{-1}(y)H_0 e^{My}, \quad H^2(y) = 0
\]

"Moduli Matrix" \( H_0 \) is a complex \( N_C \times N_F \) constant matrix

Vector multiplet BPS Eq. \( \rightarrow \) Master Eq. for Gauge Invariant \( \Omega \equiv SS^\dagger \)

\[
\partial_y (\Omega^{-1} \partial_y \Omega) = g^2 c \left( 1_C - \Omega^{-1} \Omega_0 \right), \quad \Omega_0 \equiv c^{-1} H_0 e^{2My} H_0^\dagger
\]

\( H_0 \rightarrow \Omega(y) \rightarrow \) Gauge choice \( \rightarrow S(y) \rightarrow \Sigma, W_y, H^1 \)

Bound. cond. at \( y = \pm \infty \rightarrow \) No integration const. from Master Eq for \( \Omega \)

Index Theorem, Proof of Existence and Uniqueness, \(^{[1]}\)

Complete Solution at \( g \rightarrow \infty \), Solutions at Discrete Finite Coupling,

Moduli matrix \( H_0 \) describes the entire moduli space of walls

Total Moduli Space

Global "World-Volume Symmetry" : \( N_C^2 \) Integration constants in solving \( S \)
Figure 2: A three wall solution connecting vacuum A to C through B (left). By letting the left-most wall to infinity, we obtain a two wall solution connecting vacuum A to B.

\[(S, H_0)\) and \((S', H_0')\) give the same \(H^1 = S^{-1}H_0e^{My}\) (and \(\Sigma, W_y\))

\[S \rightarrow S' = VS, \quad H_0 \rightarrow H_0' = VH_0, \quad V \in GL(N_C, C)\]

Total Moduli space for wall solutions is the complex Grassmann manifold:

\[\mathcal{M} = \{H_0|H_0 \sim VH_0, V \in GL(N_C, C)\} \equiv G_{N_F, N_C}\]

\[\sim \frac{SU(N_F)}{SU(N_C) \times SU(N_F - N_C) \times U(1)}\]

Compact (closed) set of complex dimension \(N_C \tilde{N}_C \equiv N_C(N_F - N_C)\)
\[
\mathcal{M} = \mathcal{M}^{1/1} + \mathcal{M}^{1/2} = \mathcal{M}^0 \oplus \mathcal{M}^1 \oplus \cdots \oplus \mathcal{M}^{Nc \tilde{N}c}
\]

**Effective Lagrangian on BPS Walls**

Promote the moduli parameters to fields \( \phi, \phi^* \) on the world-volume of walls

**Effective Lagrangian** is given by \( 1/2 \) BPS solution \( \Omega_{sol}(y, \phi, \phi^*) \)

\[
\mathcal{L} = -T_w + \int d^4 \theta K(\phi, \phi^*) + \text{higher derivatives}
\]

\( T_w \): Energy Density of Wall, \( K \): Kähler potential of moduli fields \( \phi, \phi^* \)

\[
K(\phi, \phi^*) = \int dy \left[ c \log \det \Omega + c \text{Tr} \left( \Omega_0 \Omega^{-1} \right) + \frac{1}{2g^2} \text{Tr} \left( \Omega^{-1} \partial_y \Omega \right)^2 \right] \bigg|_{\Omega=\Omega_{sol}}
\]

Kähler potential serves as the action for the master equation of \( \Omega \)

**Exact Solution at \( g \to \infty \): NLSM**

Strong coupling limit \( g^2 c/\Delta m \gg 1 \): BPS Eq. for \( \Omega \to \text{Algebraic equation} \)

\[
\Omega = \Omega_0 \equiv c^{-1} H_0 e^{2M_y} H_0^\dagger
\]

\( g^2 \to \infty \): Hyper-Kähler (HK) **Nonlinear Sigma Model** (NLSM)
Massless Hypermultiplet $\rightarrow$ NLSM with Target Space $T^*(G_{N_F,N_C})$
Massive Hypermultiplet: Potential $\rightarrow$ Discrete Vacua :

Moduli Space of BPS Wall Connecting Vacua : $G_{N_F,N_C}$

3 Global Structure of Wall Moduli Space

$U(1) \times U(1)$ gauge theory with FI terms $c_I, I = 1, 2$

4 hypermultiplets $h_A, A = 1, \cdots, 4$ with unequal charges

$q_I^A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -n & 1 & 1 \end{pmatrix}$

Figure 3: Hyperplane arrangement for the NLSM which contains $T^*F_n$. 

[Diagram of hyperplane arrangement]
Figure 4: Profiles of walls $\alpha_2 \to \alpha_4 \to \alpha_3 \to \alpha_1$. We plotted accumulated energy density $\int_0^y E \, dy$ in the left, and energy density $E$ in the right.

SUSY vacuum conditions:

\[
\mu_A \equiv h^{A\dagger} h_A - \tilde{h}^A \tilde{h}_A^\dagger, \quad \nu_A \equiv \tilde{h}^A h_A, \quad A = 1, \ldots, 4
\]

\[
\begin{align*}
\mu_1 + \mu_2 &= c_1, \\
- \nu_2 + \mu_3 + \mu_4 &= c_2
\end{align*}
\]

\[
\begin{align*}
\nu_1 + \nu_2 &= 0, \\
- n \nu_2 + \nu_3 + \nu_4 &= 0
\end{align*}
\]

$g \to \infty = \text{NLSM} : \text{BPS Wall Flow on Vacuum Manifold } (\mu_2, \mu_4)$
Figure 5: BPS flow in $n = 1$ case. For all cases $c_1 = (1, 1)$. From left to the right: case I) $m^A = (0, 0, 1, -1)$; case II) $m^A = (1, 0, 0, -1)$; case III) $m^A = (-1, 0, 1, 0)$. Dashed lines designate the contours of constant $m^A \mu_A$.

Figure 6: Transmutation of walls when they pass through. $m^A = (1, 0, 0, -1)$ in the left and $m^A = (2, 0, 0, -1)$ in the right.
4(hypermultiplets) − 2(Gauge constraints) = 2 freedom for BPS flows

1. Moduli space is the union of two (special Lagrangian) submanifolds
2. Repulsion and attraction of walls → middle wall position of 3 walls fixed
3. Transmutation of walls
4. Moduli space dimension can be larger than suggested by index theorem. (dim Ker$\Delta^\dagger \neq 0$)

4 Webs of Domain Walls

BPS Wall Direction ↔ Phase of the Hypermultiplet Masses
Complex Mass $\mu_A = m_A + in_A$ → Non-Parallel Walls → Wall Junctions

$\mathcal{N} = 2$ SUSY in $3 + 1$ (or lower) Dimensions

$\mathcal{L} = \text{Tr} \left[ -\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{g^2} \sum_{\alpha=1}^{2} D_\mu \Sigma_\alpha D^{\mu} \Sigma_\alpha + D_\mu H^i \left( D^{\mu} H^i \right)^\dagger \right] - V$

$V = \text{Tr} \left[ \frac{1}{g^2} \sum_{a=1}^{3} (Y^a)^2 + \sum_{\alpha=1}^{2} \left| H^i M_\alpha - \Sigma_\alpha H^i \right|^2 - \frac{1}{g^2} [\Sigma_1, \Sigma_2]^2 \right]$
\[ M_1 = \text{diag}\left(m_1, m_2, \cdots, m_{N_F}\right), \quad M_2 = \text{diag}\left(n_1, n_2, \cdots, n_{N_F}\right) \]

\[ Y^a \equiv g^2 \left( c^a 1_{N_C} - (\sigma^a)^j H^i (H^j)^\dagger \right) / 2, \quad c^a = (0, 0, c) \]

**Figure 7:** Internal structures of the domain walls with \( g\sqrt{c} \ll |\Delta m + i\Delta n| \).

Wall Junctions: \( \frac{1}{4} \) BPS States

**1/4 BPS Equations**

\[ F_{12} = i \left[ \Sigma_1, \Sigma_2 \right], \quad \mathcal{D}_1 \Sigma_2 = \mathcal{D}_2 \Sigma_1, \quad \mathcal{D}_\alpha H = H M_\alpha - \Sigma_\alpha H \]

\[ \sum_{\alpha=1,2} \mathcal{D}_\alpha \Sigma_\alpha = \frac{g^2}{2} \left( c 1_{N_C} - H H^\dagger \right) \]
Bogomol’nyi bound with $J_\alpha \equiv \text{Tr}[H(M_\alpha H^\dagger - H^\dagger \Sigma_\alpha)]$

$$\mathcal{E} \geq \mathcal{J} + \mathcal{Z}_1 + \mathcal{Z}_2 + \sum_{\alpha=1,2} \partial_\alpha J_\alpha$$

$$\mathcal{J} \equiv \frac{2}{g^2} \partial_\alpha \text{Tr} (\epsilon^{\alpha\beta} \Sigma_2 D_\beta \Sigma_1), \quad \mathcal{Z}_1 \equiv c \partial_1 \text{Tr} \Sigma_1, \quad \mathcal{Z}_2 \equiv c \partial_2 \text{Tr} \Sigma_2$$

![Diagram](image_url)

(a) Abelian junction  (b) non-Abelian junction

Figure 8: Internal structures of the junctions with $g\sqrt{c} \ll |\Delta m + i\Delta n|$  

1/4 BPS Solution:

$$H = S^{-1} H_0 e^{M_1 x^1 + M_2 x^2}, \quad W_\alpha - i \Sigma_\alpha = -i S^{-1} \partial_\alpha S, \quad \alpha = 1, 2$$

$$\partial_1 (\partial_1 \Omega \Omega^{-1}) + \partial_2 (\partial_2 \Omega \Omega^{-1}) = c g^2 (1_{NC} - \Omega_0 \Omega^{-1})$$

$$\Omega \equiv SS^\dagger, \quad \Omega_0 \equiv c^{-1} H_0 e^{2(M_1 x^1 + M_2 x^2)} H^\dagger_0$$
Total Moduli Space

\[
\mathcal{M}_{\text{tot}}^{\text{webs}} \simeq G_{N_F,N_C} = \{ H_0 \mid H_0 \sim V H_0, \ V \in GL(N_C, C) \}
\]

\[
= \mathcal{M}^{\text{webs}}_{1/4} \bigcup \mathcal{M}^{\text{walls}}_{1/2} \bigcup \mathcal{M}^{\text{vacua}}_{1/1}
\]

Abelian junction : \( Y < 0 \), Non-Abelian junction : \( Y > 0 \),
Intersection of penetrable walls : \( Y = 0 \)

Figure 9: Abelian junction with 1 loop and 3 external walls in \( N_C = 1, N_F = 4 \) model.

Grid Diagram: Rules of Construction

1. Determine masses \( \mu_A \), Plot \( N_F C_{N_C} \) vacua in complex \( \text{Tr} \Sigma \) plane.
2. Convex polygon connecting vacuum points \( \leftrightarrow \) boundary condition.
   Edge : 1/2 BPS single wall
3. Draw all possible internal segments within the convex polygon (1/2 BPS single walls) forbidding any segments to cross.

4. Triangles with $\langle \cdots A \rangle$, $\langle \cdots B \rangle$, $\langle \cdots C \rangle$: Abelian junctions, with $\langle \cdots AB \rangle$, $\langle \cdots BC \rangle$ $\langle \cdots CA \rangle$: non-Abelian junctions, Parallelograms with $\langle \cdots AB \rangle$, $\langle \cdots BC \rangle$, $\langle \cdots CD \rangle$ and $\langle \cdots DA \rangle$: intersections of two penetrable walls

Figure 10: Honeycomb web of domain walls. This web divides 37 vacua and has 18 external legs and 19 internal faces. The moduli space corresponds to $\mathbb{C}P^{36}$ whose dimension is 72.

Normalizable moduli of Web of Walls: Loops in Web
\( N_F = 4, N_C = 1 \) with \( M = \text{diag}(1, i, -1 - i, 0) \)

\[
H_0 = \sqrt{c} \left( e^{a_1+ib_1}, e^{a_2+ib_2}, e^{a_3+ib_3}, e^{a_4+ib_4} \right)
\]

\( a_j + ib_j, j = 1, 2, 3 \): external wall moduli, \( a_4 + ib_4 \): loop moduli

5 Composite Soliton of Wall, Vortex, and Monopole

Vortex can stretch between walls \( \rightarrow \) 1/4 BPS

Vortex along \( x^3 \)-axis preserves 1/2 SUSY: \( \gamma^{12} (i\sigma_3)^i_j \varepsilon_j^i = \varepsilon^i \)

Wall \( \perp x^3 \) preserves 1/2 SUSY: \( \gamma^3 (i\sigma_3)^i_j \varepsilon_j^i = \varepsilon^i \)

Automatically satisfies: \( \gamma^{123} \varepsilon^i = \varepsilon^i \) Allows Monopole in Higgs Phase

Magnetic flux is squeezed into Vortices, 1/4 SUSY is preserved

Assume: dependence on \( x^m \equiv (x^1, x^2, x^3) \) (co-dimension three)

Poincaré invariance in \( x^0, x^4 \rightarrow W_0 = W_4 = 0 \)

The 1/4 BPS equations

\[
\mathcal{D}_3 \Sigma = g^2 \left( c_1 N_C - H^1 H^{1\dagger} \right)/2 + F_{12}, \quad \mathcal{D}_3 H^1 = -\Sigma H^1 + H M, \]

\[
0 = \mathcal{D}_1 H^1 + i \mathcal{D}_2 H^1, \quad 0 = F_{23} - \mathcal{D}_1 \Sigma, \quad 0 = F_{31} - \mathcal{D}_2 \Sigma
\]

BPS bound of the energy density \( \mathcal{E} \geq t_w + t_v + t_m + \partial_m J_m \)


$t_w$, $t_v$ and $t_m$: energy densities for walls, vortices and monopoles

\[ t_w = c \partial_3 \text{Tr}(\Sigma), \quad t_v = -c \text{Tr}(F_{12}), \quad t_m = \frac{2}{g^2} \partial_m \text{Tr}(\frac{1}{2} \epsilon^{mnl} F_{nl} \Sigma) \]

Magnetic flux of monopole = $\frac{1}{2} \epsilon^{mnl} F_{nl} \Sigma$ : Projected by the Higgs field

**Solutions of 1/4 BPS Equations**

**Integrability Condition**: $[\mathcal{D}_1 + i\mathcal{D}_2, \mathcal{D}_3 + \Sigma] = 0$

Invertible complex matrix function $S(x^m) \in GL(N_C, C)$ can be defined

\[
(D_3 + \Sigma)S^{-1} = 0 \rightarrow \Sigma + iW_3 \equiv S^{-1} \partial_3 S
\]

\[
(D_1 + iD_2)S^{-1} = 0 \rightarrow W_1 + iW_2 \equiv -2iS^{-1} \bar{\partial}S
\]

where $z \equiv x^1 + ix^2$, and $\bar{\partial} \equiv \partial/\partial z^*$.  

BPS Eq. for Hypermultiplet is solved by $H^1 = S^{-1}(z, z^*, x^3) H_0(z) e^{Mx^3}$

“Moduli matrix” $H_0(z)$: $N_C \times N_F$ matrix as holomorphic functions of $z$

Master Eq. for $\Omega \equiv SS^\dagger$ with $\Omega_0 \equiv H_0 e^{2My} H_0^\dagger$

\[
4\partial(\Omega^{-1} \bar{\partial} \Omega) + \partial_3(\Omega^{-1} \partial_3 \Omega) = g^2 (c - \Omega^{-1} \Omega_0)
\]

Assuming existence of unique solution of this equation for $\Omega$,
Moduli matrix $H_0(z)$ contains complete moduli of $1/4$ BPS soliton
Moduli space: holomorphic maps from $\mathbb{C}$ to complex Grassmann manifold

$$z \rightarrow G_{N_F, N_C} = \{ H_0 \vert H_0 \simeq V H_0, \; V \in GL(N_C, \mathbb{C}) \}$$

$H_0(z) \rightarrow \Omega \rightarrow$ Gauge choice $\rightarrow S \rightarrow \Sigma, \; W_m$ and $H^1$

Shifman-Yung, Phys.Rev. D70 045004 (2004); Auzzi-Bolognesi-Evslin, hep-th/0411074; · · ·

**Exact Solutions at $g^2 \rightarrow \infty$**
BPS Eq. reduces to an algebraic eq. for $g^2 \rightarrow \infty$: $\Omega = \Omega_0 \equiv c^{-1} H_0 e^{2 My} H_0^\dagger$
Take $U(1)$ model, $(N_C = 1)$

$$H_0(z) = \sqrt{c} \left( f^1(z), \ldots, f^{N_F}(z) \right): \Omega = \sum_{A=1}^{N_F} |f^A(z)|^2 e^{2m_A x^3}$$
Walls are bent for nonconstant $f^A(z)$: zeroes $\rightarrow$ vortices

$$f^A(z) \propto (z - z_A^\alpha)^{k_A^\alpha}: \text{vorticity } k_A^\alpha \text{ at } z = z_A^\alpha \text{ on the } A\text{-th wall}$$

**Monopole in Higgs phase** is realized as a kink on vortex
Energy density of monopoles $t_m \rightarrow 0$ at $g^2 \rightarrow \infty$, but Monopole charge

$$\int_V d^3x g^2 t_m = -\pi |m_A - m_B| k \text{ for a vortex with vorticity } k$$
Figure 11: Surfaces defined by the same energy density with $t_w + t_v = 0.5c$. Vortices stretched between multi-walls: $H_0(z)e^{Mx^3} = \sqrt{c}((z - 4 - 2i)(z + 5 + 8i)e^{3/2x^3}, (z + 8 - i)(z - 7 + 6i)e^{1/2x^3 + 15/2}, z^2e^{-1/2x^3 + 15/2}, (z - 6 - 5i)(z + 6 - 7i)e^{-3/2x^3})$.

Energy density of monopole in the Higgs phase should be finite for finite $g$.

Instantons in Higgs Phase can also be realized as a vortex on vortex Calorons in the Higgs Phase $\rightarrow$ Monopoles in Higgs Phase

(Scherk-Schwarz dimensional reduction)
(a) monopole-string $\mu = 2, \, \theta = 5 \times 10^{-4}$
(b) caloron $\mu = 2, \, \theta = 0.1$
(c) caloron $\mu = 1, \, \theta = 1$
(d) instanton $\mu = 0.2, \, \theta = 5$

Figure 12: Energy density of the calorons in terms of the vortex theory.

6 Conclusion

1. The BPS Solitons are constructed in SUSY $U(N_C)$ Non-Abelian Gauge Theories in 5 dimensions with $N_F$ hypermultiplets in the fundamental representation.

2. Total moduli space of the non-Abelian walls is given by a compact complex Grassmann manifold described by the moduli matrix $H_0$

$$\mathcal{M}_{N_F,N_C} \simeq \{H_0| H_0 \sim V H_0, V \in GL(N_C, C)\}$$

$$\simeq G_{N_F,N_C} \simeq \frac{SU(N_F)}{SU(N_C) \times SU(\tilde{N}_C) \times U(1)}$$
3. **Exact solutions** of Non-Abelian Walls are obtained with full generic moduli for $g^2 \to \infty$ and with partial moduli for finite gauge coupling.

4. A general formula for the **effective Lagrangian** is obtained.

5. With a $U(1) \times U(1)$ model, we obtain transmutation of walls, and fixing of wall position by repulsion and attraction.

6. **Webs of domain walls** are obtained. There are abelian and non-abelian junctions of walls in non-abelian gauge theory. Normalizable moduli of the web of walls are associated with loops of walls.

7. Moduli space of a 1/4 BPS equation is also obtained for composite configurations made of walls, vortices and monopoles in the Higgs phase, assuming existence and uniqueness of the solution of the master equation.

8. All possible 1/4 BPS solutions are obtained exactly and explicitly in the strong coupling limit.

9. **Instantons in the Higgs phase** can be realized as a lump on a vortex.
References of Solitons in 8 SUSY Theories

Tokyo Tech Collaboration

1. Domain Walls in 5D Supersymmetric Theories
   “Moduli space of BPS walls in supersymmetric gauge theories”, hep-th/0503136,
   “Global structure of moduli space for BPS walls”,
   hep-th/0503033, Phys.Rev. D71 (2005) 105009,
   “D-brane Construction for Non-Abelian Walls”, hep-th/0412024,
   Phys.Rev. D71, 125006 (2005),
   “Non-Abelian Walls in Supersymmetric Gauge Theories”,
   “Construction of Non-Abelian Walls and Their Complete Moduli Space”,
   “Exact Wall Solutions in 5-Dimensional SUSY QED”,
   hep-th/0310189, JHEP 11 (2003) 060,
   “Massless Localized Vector Field on a Wall in Five Dimensions”,
   hep-th/0310130, JHEP 11 (2003) 061,
   “Vacua of Massive Hyper-Kähler Sigma Models with Non-Abelian Quotient”,
   hep-th/0307274, Prog. Theor. Phys. 113 (2005) 657,
   “Manifest Supersymmetry for BPS walls in $\mathcal{N} = 2$ nonlinear sigma models”,
   “BPS Wall in $\mathcal{N} = 2$ SUSY Nonlinear Sigma Model with Eguchi-Hanson Manifold”,

22
hep-th/0302028, in “Garden of Quanta” - In honor of Hiroshi Ezawa, pages 299-325,

2. \(1/4\) BPS states
   “Non-Abelian Webs of Walls”, hep-th/0508241, Phys.Lett.to appear,
   “All Exact Solutions of a \(1/4\) Bogomol’nyi-Prasad-Sommerfield Equation”, hep-th/0405129, Phys.Rev.D\textbf{71}, 065018 (2005),
   “BPS Lumps and Their Intersections in \(\mathcal{N} = 2\) SUSY Nonlinear Sigma Models”, hep-th/0108133, Grav.Cosmol.\textbf{8} (2002) 129-137,

3. Vortex

4. Non-BPS Walls and Supersymmetry Breaking

5. Wall Solution in Supergravity
Solitons in 8 SUSY Theories

1. Early works

2. Wall Solution in 8 SUSY Models
   J. P. Gauntlett, D. Tong, and P. K. Townsend, Phys.Rev. D 63, 085001 (2001);
   J. P. Gauntlett, R. Portugues, D. Tong, P. K. Townsend, Phys.Rev. D 63, 085002 (2001);

3. Monopoles in Higgs Phase

4. Vortex

5. Brane construction
6. Index theorem

Other results related to the Brane-World

1. Walls in 5D SUGRA $\rightarrow$ Warped metric models
   Consistent solution of field eq. and Einstein eq.

2. Non-BPS multi-Walls $\rightarrow$ Models of SUSY Breaking

3. $U(1)$ gauge field Localization on walls with tensor multiplet
   Isozumi-Ohashi-Sakai, JHEP \textbf{11} (2003) 061; $\cdots$,
No Moduli from Vector Multiplet BPS Equation

Index Theorem

$U(1)$ gauge theory with $N_F$ flavors

$U(N_C)$ gauge theory with $N_F (> N_C)$ flavors

$$\dim_{\mathbb{C}} \text{Ker} \Delta - \dim_{\mathbb{C}} \text{Ker} \Delta^\dagger = N_C(N_F - N_C)$$

$\Delta$ is the differential operator for fluctuations


N.Sakai and D.Tong, JHEP 03, 019 (2005),

Existence and Uniqueness of the Solution

Master eq. for $\psi \equiv -\frac{1}{2} \log \Omega$ in $U(N_C = 1)$ with $N_F$ Flavors

$$\partial_y^2 \psi(y) = c g^2 \left( \sum_{A=1}^{N_F} e^{2\psi(y) + 2m_A(y - Y_0) + 2r_A} - 1 \right)$$

“Positive definite” kernel $\rightarrow$ Existence and Uniqueness of the solution

N.Sakai and Y.Yang, hep-th/0505136

moduli
Figure 13: Total moduli space of $N_F = 3$, $U(1)$ model, which is the same as $N_F = 3$, $U(2)$ model by duality. The sum $\mathcal{M}^2_{3,1} \oplus \mathcal{M}^1_{3,1} \oplus \mathcal{M}^0_{3,1}$ is decomposed. The double-wall and single-wall sectors are non-compact and the zero-wall (vacuum) sector is compact.

Moduli Space Decomposition

total-moduli-sp
Interaction Rules of Walls, Vortices, Monopoles

1. Vortex (without monopole) can only connect between impenetrable pairs of walls.

2. Vortex can end on a wall giving a binding energy (Boojum).

3. Vortex with monopole can connects a penetrable pair of walls to make them impenetrable.

4. Vortex can penetrate through a wall if the vortex has no flavor quantum number in common with the wall.
Figure 14: For the $\alpha_i$ elementary domain wall, the $q_i$ string may end on the left and the $q_{i+1}$ string may end on the right. All $q_j$ strings, for $j \neq i, i + 1$ exist in both left and right vacua and pass right through the domain wall. The nodes represent the finite binding energy.