Solitons in the Higgs Phase: Moduli Matrix Approach

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1 Discrete Vacua in Higgs Phase and BPS Eq.

Brane-World = Our world on a Topological Defect in higher dim. spacetime

Topological Defects : Walls, Junctions, . . . are preferably Solitons

Part of Supersymmetry (SUSY) preserved → BPS state

Solves the Equation of Motion

Soliton Dynamics: Important for Nonperturbative Effects

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**SUSY** $U(N_C)$ Gauge Theory with $N_F$ Flavors

5 Dimensions → 8 SUSY $(M, N, \cdots = 0, 1, 2, 3, 4)$

Vector multiplets: $W_M$ Gauge field, $\Sigma$ Real Scalar ($N_C \times N_C$ matrix)

Hypermultiplets: $(H^i)^r_A \equiv H^{irA}$ Complex Scalar ($N_C \times N_F$ matrix)
$(i = 1, 2; \text{Color } r = 1, \cdots, N_C; \text{Flavor } A = 1, \cdots, N_F)$

Minimal kinetic terms

Gauge coupling $g$ for $U(N_C)$, Fayet-Iliopoulos (FI) parameter $c$

Hypermultiplet Mass $(M)^A_B \equiv m_A \delta^A_B$

$$
\mathcal{L} = -\frac{1}{2g^2} \text{Tr}(F_{MN}(W)F^{MN}(W)) + \frac{1}{g^2} \text{Tr}(\mathcal{D}^M \Sigma \mathcal{D}_M \Sigma) \\
+ \text{Tr} [\mathcal{D}^M H^i (\mathcal{D}_M H^i)^\dagger] - V
$$

$$
V = \frac{g^2}{4} \text{Tr} \left[ (H^1 H^{1\dagger} - H^2 H^{2\dagger} - c 1_{N_C})^2 + 4H^2 H^{1\dagger} H^1 H^{2\dagger} \right] \\
+ \text{Tr} \left[ (\Sigma H^i - H^i M)(\Sigma H^i - H^i M)^\dagger \right]
$$

Non-degenerate mass: $m_A > m_{A+1} \rightarrow$ Flavor symmetry: $U(1)^{N_F-1}_F$

**Discrete SUSY Vacua**: Color-flavor locking $\langle A_1 A_2 \cdots A_{N_C} \rangle$

$$
H^{1rA} = \sqrt{c} \delta^{A_r}_A, \quad H^{2rA} = 0, \quad \Sigma = \text{diag}(m_{A_1}, \cdots, m_{A_{N_C}})
$$

Number: $\frac{N_F!}{(N_F-N_C)! N_C!} \sim e^{N_F \log(x^{-x}(1-x)^{-(1-x)})}, \quad x \equiv N_C/N_F$

**Higgs Phase**: Walls and Vortices as elementary Solitons
2 1/2 BPS Walls

1/2 BPS Equations

Dependence on $y \equiv x^4$, 4 D Poincaré invariance $\rightarrow W_M \neq y = 0$

Bogomol’nyi completion of energy density $\mathcal{E}$

$$\mathcal{E} = \frac{g^2}{4} \text{Tr} \left[ \left( c_{1N_C} - H^1 H^1 \dagger + H^2 H^2 \dagger - \frac{2}{g^2} D_y \Sigma \right)^2 + 4 H^2 H^1 \dagger H^1 H^2 \dagger \right]$$

$$+ \text{Tr} D_y H^1 + \Sigma H^1 - H^1 M \right)^2 + \text{Tr} D_y H^2 - \Sigma H^2 + H^2 M \right)^2$$

$$+ c \partial_y \text{Tr} \Sigma - \partial_y \left\{ \text{Tr} \left[ (\Sigma H^1 - H^1 M) H^1 \dagger + (-\Sigma H^2 + H^2 M) H^2 \dagger \right] \right\}$$

1/2 BPS equations $\iff$ Conserved SUSY: $\gamma^4 \varepsilon^i = -i (\sigma^3)^{i \ j} \varepsilon^j$

$$D_y \Sigma = g^2 \left( c_{1N_C} - H^1 H^1 \dagger + H^2 H^2 \dagger \right) / 2, \quad 0 = g^2 H^1 H^2 \dagger$$

$$D_y H^1 = -\Sigma H^1 + H^1 M, \quad D_y H^2 = \Sigma H^2 - H^2 M$$

Boundary condition: Vacuum at $y = -\infty$ and at $y = +\infty$

Solving BPS Equations

$S(y) \in GL(N_C, \mathbb{C})$ defined by: $\Sigma + i W_y \equiv S^{-1}(y) \partial_y S(y)$

BPS Eqs. for Hypermultiplet can be solved by

$$H^1(y) = S^{-1}(y) H_0 e^{M y}, \quad H^2(y) = 0$$
“Moduli Matrix” $H_0$ is a complex $N_C \times N_F$ constant matrix
Vector multiplet BPS Eq. $\rightarrow$ Master Eq. for Gauge Invariant $\Omega \equiv SS^\dagger$

$$\partial_y \left( \Omega^{-1} \partial_y \Omega \right) = g^2 c \left( 1_C - \Omega^{-1} \Omega_0 \right), \quad \Omega_0 \equiv c^{-1} H_0 e^{2My} H_0^\dagger$$

$H_0 \rightarrow \Omega(y) \rightarrow$ Gauge choice $\rightarrow S(y) \rightarrow \Sigma, W_y, H^1$

Given $H_0 \rightarrow$ Unique solution exists for $\Omega$ (No new integration const.)

Boundary conditions at $y = \pm \infty$: encoded in $H_0$

Index Theorem, Existence and Uniqueness Proof ($U(1)$) [index theorem]

Complete Solution at $g \rightarrow \infty$, Solutions at Discrete Finite Coupling,

Moduli matrix $H_0$ describes the moduli space of solutions of BPS eq.
Figure 2: A three wall solution connecting vacuum A to C through B (left). By letting the left-most wall to infinity, we obtain a two wall solution connecting vacuum A to B.

**Topology of Total Moduli Space**

“\(V\)-equivalence Relation” : \(N_C^2\) Integration constants in solving \(S\)

\((S, H_0)\) and \((S', H_0')\) give the same \(H^1 = S^{-1}H_0 e^{M_y}\) (and \(\Sigma, W_y\))

\[ S \rightarrow S' = V S, \quad H_0 \rightarrow H_0' = VH_0, \quad V \in GL(N_C, C) \]

Total Moduli space for solutions of BPS eq. : complex Grassmann manifold

\[ \mathcal{M} = \{ H_0 | H_0 \sim VH_0, V \in GL(N_C, C) \} \equiv G_{N_F,N_C} \]

\[ \cong \frac{SU(N_F)}{SU(N_C) \times SU(N_F - N_C) \times U(1)} \]
The \( n - 1 \)-walls sector as Boundaries of a \( n \)-walls sector

\[
\mathcal{M} = \mathcal{M}^{1/1} + \mathcal{M}^{1/2} = \mathcal{M}^0 \oplus \mathcal{M}^1 \oplus \cdots \oplus \mathcal{M}^{N_C(N_F-N_C)}
\]

Effective Lagrangian on BPS Walls

Promote the moduli parameters to fields \( \phi, \phi^* \) on the world-volume of walls

Effective Lagrangian is given by 1/2 BPS solution \( \Omega_{\text{sol}}(y, \phi, \phi^*) \)

\[
\mathcal{L} = -T_w + \int d^4\theta K(\phi, \phi^*) + \text{higher derivatives}
\]

\( T_w \): Energy Density of Wall, \( K \): Kähler potential of moduli fields \( \phi, \phi^* \)

\[
K(\phi, \phi^*) = \int dy \left[ c \log \det \Omega + c \text{Tr} (\Omega_0 \Omega^{-1}) + \frac{1}{2g^2} \text{Tr} (\Omega^{-1} \partial_y \Omega)^2 \right] \bigg|_{\Omega=\Omega_{\text{sol}}}
\]

Kähler potential serves as the action for the master equation of \( \Omega \)

Exact Solution at \( g \to \infty \): NLSigma Model

Strong coupling limit \( g^2 c/\Delta m \gg 1 \): BPS Eq. for \( \Omega \to \) Algebraic equation

\[
\Omega = \Omega_0 \equiv c^{-1} H_0 e^{2M_y H_0^\dagger}
\]
\( g^2 \to \infty \): Hyper-Kähler (HK) **Nonlinear Sigma Model** (NLSM)

Massless Hypermultiplet: NLSM with Target Space \( T^* (G_{N_F,N_C}) \)

Massive Hypermultiplet: with Potential \( \to \) Discrete Vacua:

- **Moduli Space** of Solutions of the BPS Equations: \( G_{N_F,N_C} \)
- \[ \dim(G_{N_F,N_C}) = \frac{1}{2} \dim(T^* (G_{N_F,N_C})) : (H^1 \neq 0, H^2 = 0) \]

## 3 Composite Solitons: Webs of Domain Walls

BPS Wall Direction \( \leftrightarrow \) Phase of the Hypermultiplet Masses

(Relatively) **Complex Masses** \( \mu_A = m_A + i n_A \to \) Non-Parallel Walls

\( \to \) **Wall Junctions**: \( 1/4 \) BPS States

\( \mathcal{N} = 2 \) SUSY in \( 3 + 1 \) (or lower) Dimensions

\[
\mathcal{L} = \text{Tr} \left[ -\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{g^2} \sum_{\alpha=1}^{2} D_\mu \Sigma_\alpha D^\mu \Sigma_\alpha + D_\mu H^i (D^\mu H^i)^\dagger \right] - V
\]

\[
V = \text{Tr} \left[ \frac{1}{g^2} \sum_{a=1}^{3} (Y^a)^2 + \sum_{\alpha=1}^{2} |H^i M_\alpha - \Sigma_\alpha H^i|^2 - \frac{1}{g^2} [\Sigma_1, \Sigma_2]^2 \right]
\]

\[
M_1 = \text{diag} (m_1, m_2, \cdots , m_{N_F}) , \quad M_2 = \text{diag} (n_1, n_2, \cdots , n_{N_F})
\]
\[ Y^a \equiv g^2 \left( c^a 1_{NC} - (\sigma^a)^j i H^i (H^j)^\dagger \right) / 2, \quad c^a = (0, 0, c) \]

(a) Abelian gauge theory

(b) non-Abelian gauge theory

Figure 3: Internal structures of the domain walls with \( g\sqrt{c} \ll |\Delta m + i\Delta n| \).

1/4 BPS Equations:

\[ \Gamma_w = \gamma^1 \otimes i\sigma^3, \quad \Gamma_{w'} = -i\gamma^2\gamma^5 \otimes i\sigma^3 \]

Inv. under \((z, \Sigma, M) \rightarrow e^{i\theta} (z, \Sigma, M), \quad (\lambda^i, \psi) \rightarrow e^{-i\theta^2 \gamma^5 - \theta^2 \gamma^{12}} (\lambda^i, \psi)\]

\[ F_{12} = i [\Sigma_1, \Sigma_2], \quad D_1 \Sigma_2 = D_2 \Sigma_1 \]

\[ D_\alpha H = HM_\alpha - \Sigma_\alpha H \quad (3.2) \]

\[ \sum_{\alpha=1,2} D_\alpha \Sigma_\alpha = \frac{g^2}{2} (c1_{NC} - HH^\dagger) \quad (3.3) \]
Bogomol’nyi bound \((J_\alpha \equiv \text{Tr}\left[H(M_\alpha H^\dagger - H^\dagger \Sigma_\alpha)\right])\)

\[
\mathcal{E} \geq \mathcal{Y} + \mathcal{Z}_1 + \mathcal{Z}_2 + \sum_{\alpha=1,2} \partial_\alpha J_\alpha
\]

\[
\mathcal{Y} \equiv \frac{2}{g^2} \partial_\alpha \text{Tr} \left( \epsilon^{\alpha\beta} \Sigma_2 \mathcal{D}_\beta \Sigma_1 \right), \quad \mathcal{Z}_1 \equiv c \partial_1 \text{Tr} \Sigma_1, \quad \mathcal{Z}_2 \equiv c \partial_2 \text{Tr} \Sigma_2
\]

(a) Abelian junction \quad (b) non-Abelian junction

Figure 4: Internal structures of the junctions with \(g \sqrt{c} \ll |\Delta m + i \Delta n|\)

1/4 BPS Solution: (3.2) is solved by \(S\), (integrability cond (3.1))

\[
H = S^{-1} H_0 e^{M_1 x^1 + M_2 x^2}, \quad \mathcal{W}_\alpha - i \Sigma_\alpha = -i S^{-1} \partial_\alpha S, \quad \alpha = 1, 2
\]

\[
\partial_1 \left( \partial_1 \Omega \Omega^{-1} \right) + \partial_2 \left( \partial_2 \Omega \Omega^{-1} \right) = cg^2 \left( \mathbb{1}_{NC} - \Omega_0 \Omega^{-1} \right)
\]

\[
\Omega \equiv SS^\dagger, \quad \Omega_0 \equiv c^{-1} H_0 e^{2(M_1 x^1 + M_2 x^2)} H_0^\dagger
\]
Total Moduli Space

\[ \mathcal{M}_{\text{tot webs}} \simeq G_{N_F,N_C} = \{ H_0 \mid H_0 \sim V H_0, \ V \in GL(N_C, C) \} \]
\[ = \mathcal{M}_{\text{webs}}^{1/4} \cup \mathcal{M}_{\text{walls}}^{1/2} \cup \mathcal{M}_{\text{vacua}}^{1/1} \]

Abelian junction : \( Y < 0 \), Non-Abelian junction (Hitchin Vortex): \( Y > 0 \),
Intersection of penetrable walls : \( Y = 0 \)

Figure 5: Abelian junction with 1 loop and 3 external walls in \( N_C = 1, \ N_F = 4 \) model.

Grid Diagram is a useful tool

Normalizable moduli of Web of Walls: Loops in Web

\( N_F = 4, \ N_C = 1 \) with \( M = \text{diag}(1, i, -1 - i, 0) \)

\[ H_0 = \sqrt{c} \left( e^{a_1 + ib_1}, e^{a_2 + ib_2}, e^{a_3 + ib_3}, e^{a_4 + ib_4} \right) \]

\( a_j + ib_j, j = 1, 2, 3 \): external wall moduli, \( a_4 + ib_4 \): loop moduli
Figure 6: Honeycomb web of domain walls. This web divides 37 vacua and has 18 external legs and 19 internal faces. The moduli space corresponds to $\mathbb{C}P^{36}$ whose dimension is 72.

4 Composite Soliton: Monopole in Higgs Phase

Monopole (and Instanton) is $1/2$ BPS state in Coulomb (unbroken) phase

They cannot exist alone in the Higgs Phase

Magnetic flux is squeezed in Higgs phase $\rightarrow$ Monopole with Vortices

Monopole preserves $1/2$ SUSY: $\gamma^{123} \epsilon^i = \epsilon^i$

Vortex along $x^3$-axis preserves $1/2$ SUSY: $\gamma^{12} (i\sigma_3)^i j \epsilon^j = \epsilon^i$
Automatically satisfies: $\gamma^3 (i\sigma_3)^i_j \varepsilon^j = \varepsilon^i$ Allows $\text{Wall } \perp x^3$

$\rightarrow 1/4 \text{ BPS Composite States of } \text{Walls, Vortices, and Monopoles}$

Assume: dependence on $x^m \equiv (x^1, x^2, x^3)$ (co-dimension three)

Poincaré invariance in $x^0, x^4 \rightarrow W_0 = W_4 = 0$

**The 1/4 BPS equations**

$$0 = F_{23} - \mathcal{D}_1 \Sigma, \quad 0 = F_{31} - \mathcal{D}_2 \Sigma \quad (4.1)$$

Contribution of vortex magnetic field $F_{12}$ added to Wall BPS Eq.

$$\mathcal{D}_3 H^1 = -\Sigma H^1 + HM, \quad 0 = \mathcal{D}_1 H^1 + i\mathcal{D}_2 H^1$$

$$\mathcal{D}_3 \Sigma = g^2 \left( c_1 N_C - H^1 H^1\dagger \right) / 2 + F_{12}$$
**BPS bound** of the energy density $\mathcal{E} \geq t_w + t_v + t_m + \partial_m J_m$

$t_w$, $t_v$ and $t_m$: energy densities for **walls**, **vortices** and **monopoles**

$t_w = c \partial_3 \text{Tr}(\Sigma)$,  
$t_v = -c \text{Tr}(F_{12})$,  
$t_m = \frac{2}{g^2} \partial_m \text{Tr}(\frac{1}{2} \epsilon^{mn} F_{nl} \Sigma)$

Magnetic flux of monopole $= \frac{1}{2} \epsilon^{mn} F_{nl} \Sigma$ : Projected by the Higgs field

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**Solutions of 1/4 BPS Equations**

(4.1) guarantees the **Integrability Condition** : $[\mathcal{D}_1 + i\mathcal{D}_2, \mathcal{D}_3 + \Sigma] = 0$

Invertible complex matrix function $S(x^m) \in GL(N_C, \mathbb{C})$ can be defined

$$(\mathcal{D}_3 + \Sigma) S^{-1} = 0 \rightarrow \Sigma + iW_3 \equiv S^{-1} \partial_3 S$$

$$(\mathcal{D}_1 + i\mathcal{D}_2) S^{-1} = 0 \rightarrow W_1 + iW_2 \equiv -2i S^{-1} \bar{\partial} S$$

where $z \equiv x^1 + ix^2$, and $\bar{\partial} \equiv \partial / \partial z^*$.

BPS Eq. for Hypermultiplet is solved by $H_1 = S^{-1}(z, z^*, x^3) H_0(z) e^{Mx^3}$

"Moduli matrix" $H_0(z)$: $N_C \times N_F$ matrix as **holomorphic functions** of $z$

Master Eq. for $\Omega \equiv S S^\dagger$ with $\Omega_0 \equiv H_0 e^{2My} H_0^\dagger$

$$4\partial(\Omega^{-1} \bar{\partial} \Omega) + \partial_3(\Omega^{-1} \partial_3 \Omega) = g^2 (c - \Omega^{-1} \Omega_0)$$
Assuming existence of unique solution of this equation for $\Omega$,

Moduli matrix $H_0(z)$ contains complete moduli of $1/4$ BPS soliton


Figure 8: Surfaces defined by the same energy density with $t_w + t_v = 0.5c$. Vortices streched between multi-walls: $H_0(z)e^{My}H_0^\dagger = \sqrt{c}((z - 4 - 2i)(z + 5 + 8i)e^{3/2x^3}, (z + 8 - i)(z - 7 + 6i)e^{1/2x^3 + 15/2}, (z - 6 - 5i)(z + 6 - 7i)e^{-3/2x^3})$.

Exact Solutions at $g^2 \to \infty$

BPS Eq. reduces to an algebraic eq. for $g^2 \to \infty$: $\Omega = \Omega_0 \equiv c^{-1}H_0e^{2My}H_0^\dagger$
Take $\mathbf{U}(1)$ model, ($N_C = 1$)

$$H_0(z) = \sqrt{c} \left( f^1(z), \ldots, f^{N_F}(z) \right): \Omega = \sum_{A=1}^{N_F} |f^A(z)|^2 e^{2m_A x^3}$$

Walls are bent for nonconstant $f^A(z)$: zeroes $\rightarrow$ vortices

$$f^A(z) \propto (z - z^A_\alpha)^k^A_\alpha: \text{vorticity } k^A_\alpha \text{ at } z = z^A_\alpha \text{ on the } A\text{-th wall}$$

Monopole in Higgs phase $= \text{a kink on vortex}$ in Non-Abelian case

Energy density of monopole in the Higgs phase should be finite for finite $g$

Instantons in Higgs Phase can also be realized as a lump on vortex

5 Conclusion

1. The BPS Solitons are constructed in SUSY $\mathbf{U}(N_C)$ Non-Abelian Gauge Theories in 5 dimensions with $N_F$ hypermultiplets in the fundamental representation.
2. **Total moduli space** of the non-Abelian walls is given by a compact complex Grassmann manifold described by the moduli matrix $H_0$

\[ \mathcal{M}_{N_F, N_C} \cong \{ H_0 | H_0 \sim V H_0, V \in GL(N_C, C) \} \]

\[ \cong G_{N_F, N_C} \cong \frac{SU(N_F)}{SU(N_C) \times SU(\tilde{N}_C) \times U(1)} \]

3. **Exact solutions** of solitons are obtained for $g^2 \to \infty$.

4. A general formula for the effective Lagrangian is obtained.

5. **Webs of domain walls** are obtained. There are abelian and non-abelian junctions of walls in non-abelian gauge theory. Normalizable moduli of the web of walls are associated with loops of walls.

6. Moduli space of a $1/4$ BPS equation is obtained for composite configurations made of walls, vortices and monopoles in the Higgs phase, assuming existence and uniqueness of the solution of the master equation.

7. Instantons in the Higgs phase can be realized as a lump on a vortex.
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Tokyo Tech Collaboration

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**Solitons in 8 SUSY Theories**

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3. Monopoles in Higgs Phase

4. Vortex

5. Brane construction  

6. Index theorem  
Other results related to the Brane-World

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2. Non-BPS multi-Walls $\rightarrow$ Models of SUSY Breaking


3. $\boldsymbol{U(1)}$ gauge field Localization on walls with tensor multiplet

   Isozumi-Ohashi-Sakai, JHEP $\mathbf{11}$ (2003) 061; ···,
No Moduli from Vector Multiplet BPS Equation

Index Theorem

$U(1)$ gauge theory with $N_F$ flavors

$U(N_C)$ gauge theory with $N_F(> N_C)$ flavors

$$\dim \mathbb{C} \ker \Delta - \dim \mathbb{C} \ker \Delta^\dagger = N_C(N_F - N_C)$$

$\Delta$ is the differential operator for fluctuations


N.Sakai and D.Tong, JHEP 03, 019 (2005),

Existence and Uniqueness of the Solution

Master eq. for $\psi \equiv -\frac{1}{2} \log \Omega$ in $U(N_C = 1)$ with $N_F$ Flavors

$$\partial_y^2 \psi(y) = c g^2 \left( \sum_{A=1}^{N_F} e^{2\psi(y) + 2m_A(y-Y_0) + 2r_A} - 1 \right)$$

“Positive definite” kernel $\rightarrow$ Existence and Uniqueness of the solution

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Figure 10: Total moduli space of $N_F = 3$, $U(1)$ model, which is the same as $N_F = 3$, $U(2)$ model by duality. The sum $\mathcal{M}^2_{3,1} \oplus \mathcal{M}^1_{3,1} \oplus \mathcal{M}^0_{3,1}$ is decomposed. The double-wall and single-wall sectors are non-compact and the zero-wall (vacuum) sector is compact.

Moduli Space Decomposition
Global Structure of Wall Moduli Space

\( U(1) \times U(1) \) gauge theory with FI terms \( c_I, I = 1, 2 \)

4 hypermultiplets \( H^1_A, H^2_A, A = 1, \ldots, 4 \) with unequal charges

\[
q^A_I = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & -n & 1 & 1 \\
\end{pmatrix}
\]

SUSY vacuum conditions:

\[
\mu_A \equiv H_A^{1\dagger} H^1_A - H_A^{2\dagger} H^2_A, \quad \nu_A \equiv H_A^2 H_A^1, \quad A = 1, \ldots, 4 \\
\mu_1 + \mu_2 = c_1, \quad -n \mu_2 + \mu_3 + \mu_4 = c_2
\]
Figure 12: Profiles of walls $\alpha_2 \rightarrow \alpha_4 \rightarrow \alpha_3 \rightarrow \alpha_1$. We plotted accumulated energy density $\int_0^y \mathcal{E} \, dy$ in the left, and energy density $\mathcal{E}$ in the right.

$$\nu_1 + \nu_2 = 0, \quad -n\nu_2 + \nu_3 + \nu_4 = 0$$

$g \rightarrow \infty = \text{NLSM : BPS Wall Flow on Vacuum Manifold (}\mu_2, \mu_4)\text{)}$

4(hypermultiplets)−2(Gauge constraints)=2 freedom for BPS flows

1. Moduli space is the union of two (special Lagrangian) submanifolds
2. Repulsion and attraction of walls $\rightarrow$ middle wall position of 3 walls fixed
Figure 13: BPS flow in $n = 1$ case. For all cases $c_I = (1,1)$. From left to the right: case I) $m^A = (0,0,1,-1)$; case II) $m^A = (1,0,0,-1)$; case III) $m^A = (-1,0,1,0)$. Dashed lines designate the contours of constant $m^A \mu_A$.

Figure 14: Transmutation of walls when they pass through. $m^A = (1,0,0,-1)$ in the left and $m^A = (2,0,0,-1)$ in the right.
3. Transmutation of walls

4. Moduli space dimension can be larger than suggested by index theorem. 
   \((\dim \operatorname{Ker} \Delta^\dagger \neq 0)\)